Computability and Complexity Theory

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You are encouraged to work together on solving homework problems,
but please put their names clearly at the top of the assignment.
Everyone must turn in their own independently written solutions.
The homework is due by the beginning of class (strict).
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Homework 2, due May 22

1. Prove that there does not exist any first-order formula $Z(x)$ in the language $\langle 0, +, \leq \rangle$ that over the field $\mathbb{Q}$ of rational numbers expresses the fact that x is an integer.

2. Show[1] that the word problem in the group $\langle x, y || x^{1984} = y^{2014} = 1 \rangle$ is decidable.

3. Prove that there is no algorithm that for any given system

$$p_1(x_1, \ldots, x_n) = 0, \ldots, p_m(x_1, \ldots, x_n) = 0$$

of diophantine equations with the property that **the degree of every equation is odd** tells if this system has an integer solution or not. You may refer to Matiyasevich's theorem, of course.

4. Let $S \stackrel{\mathrm{def}}{=} \{0,1\}^n$ be the probability space with uniform distribution, and let $X_k : S \longrightarrow \mathbb{Z}_k$ count the number of ones in the input $\mod k$. Which of the two quantities $I(X_{2014} : X_2)$ and $I(X_{1984} : X_4)$ is larger and why?

[1]The standards of mathematical rigor will be significantly relaxed for this problem: a correct algorithm, along with some intuition as to why it is correct will pass for a solution.

5. A set A is *i.o. lean* ("i.o" stands for "infinitely often") if

$$\liminf_{n\to\infty} \frac{|A^{\leq n}|}{n} = 0$$

(note the $\liminf$!) Prove that there are i.o. lean recursive sets A that contain infinitely many incompressible strings.