Computability and Complexity Theory

Instructor: Alexander Razborov, University of Chicago
razborov@cs.uchicago.edu
Course Homepage: www.cs.uchicago.edu/~razborov/teaching/spring14.html

Spring Quarter, 2014

`You are encouraged to work together on solving homework problems, but please put their names clearly at the top of the assignment. Everyone must turn in their own independently written solutions.`

Homework 3, due June 4

1. Let $\omega = \omega_1\omega_2 \ldots \omega_n \ldots$ be an infinite random sequence (w.r.t. the uniform measure), and let the Boolean function $MAJ_k(x_1, \ldots, x_k)$ output 1 if and only if at least $\lceil k/2 \rceil$ of its inputs are one. Which of the two sequences $MAJ_3(\omega_1, \omega_2, \omega_3)MAJ_3(\omega_4, \omega_5, \omega_6)MAJ_3(\omega_7, \omega_8, \omega_9)\ldots$ and $MAJ_4(\omega_1, \omega_2, \omega_3, \omega_4)MAJ_4(\omega_5, \omega_6, \omega_7, \omega_8)MAJ_4(\omega_9, \omega_{10}, \omega_{11}, \omega_{12})\ldots$ is also random and why.

   `Note.` I do not expect excessive rigor here in terms of "intuitively obvious" facts from measure theory.

2. Let $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ be a poly-time computable function such that for all x, y such that $y \in \{0, x\}$, $f(x, y) = x + y$. Prove that there exists another poly-time computable function $g(x)$ such that $f(x, g(x)) \leq f(x, g(x) + 1)$ for all $x > 0$.

   `Note.` Numbers are represented in binary.

3. Construct an *explicit* and *direct* (that is, bypassing the Cook-Levin theorem) poly-time Karp reduction from VERTEX COVER to SATISFIABILITY.

4. Prove that the following problem is **NP**-complete:

   `INPUT.` A set of vectors $v_1, \ldots, v_r \in \mathbb{N}^d$ and a positive integer k.

`QUESTION.` Does there exist a subset $S \subseteq \{1, \ldots, r\}$ of cardinality k such that all coordinates in the vector $\sum_{i \in S} v_i$ are powers of two?