

Computability and Complexity Theory

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Course Homepage: www.cs.uchicago.edu/~razborov/teaching/winter17.html

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You may work together on solving homework problems, but please put all the names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. If you find any of the solutions on-line, it is OK but then you must completely understand the proof, explain it in your own words and include the URL. The due date below pertains to e-mail submissions as a PDF file prepared from a TeX source: this is the preferred format. Handwritten work must be submitted by the beginning of class a day before the deadline.

Homework 1, due February 8

1. Prove or disprove that:
 - (a) every infinite r.e. set can be represented as $\text{im}(f)$ for an *injective* total recursive function f ;
 - (b) every infinite recursive set can be represented as $\text{im}(f)$ for an injective *primitive* recursive function f .
2. Design a Markov algorithm for computing the characteristic function of the (binary) equality predicate. In other words, the algorithm must transform a string of the form $1^x \# 1^y$ into 1 if $x = y$ and into Λ otherwise.
3. An URM M *primarily accepts* an input x if in the course of its computation on x , 0 appears in the register R_1 only finitely many times, and, moreover, this number is prime. Let

$$L_{\text{prime}} \stackrel{\text{def}}{=} \{(x, y) \mid M_x \text{ primarily accepts } y\}$$

(M_x is the URM with code x).

- (a) Prove that every r.e. set is many-one reducible to L_{prime} .
 - (b) Prove that every r.e. set is many-one reducible to its complement $\overline{L_{\text{prime}}}$.
 - (c) Prove that L_{prime} is not r.e.
4. Let us call a r.e. set A *atomic* if it is infinite but for any r.e. set B such that $B \subseteq A$, either B is finite or $A \setminus B$ is finite. Prove that atomic sets do not exist¹.
5. Let G be a fixed *recursive graph* on \mathbb{N} , i.e. a recursive subset $E = E(G) \subseteq \mathbb{N} \times \mathbb{N}$ such that $\forall x \in \mathbb{N} (\langle x, x \rangle \notin E)$ and $\forall x, y \in \mathbb{N} (\langle x, y \rangle \in E \equiv \langle y, x \rangle \in E)$. Recall that $Z(x)$ is the identically zero function in one argument.
- (a) Come up with a definition of an operator $\eta : \mathcal{F}_2 \longrightarrow \mathcal{F}_1$, also written as $g(x, y) \mapsto \eta g(x, y) = 0$ with the following two properties:
 - i. η takes recursive functions to recursive functions;
 - ii. $(\eta g)(x)$ is defined if and only if there exists y such that $g(x, y) = 0$, and in that case $g(x, (\eta g)(x)) = 0$.
 - (b) Explain why such an operator can never be recursive. Explain why your construction does not contradict Myhill-Shepherdson.
 - (c) Which of the following operators on \mathcal{F}_2 :

$$\Phi_0(f)(x, y) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } (x, y) \in E \\ \eta z(f(x, z) + f(y, z) \geq 0) & \text{otherwise} \end{cases}$$

$$\Phi_1(f)(x, y) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } (x, y) \in E \\ Z(\eta z(f(x, z) + f(y, z) \geq 0)) & \text{otherwise} \end{cases}$$

is recursive and why?

- (d) Compute the least fixed point of this operator (the answer should be given in graph-theoretical terms).

¹I do not immediately see whether it is true for coatomic r.e. sets, defined (in the obvious way) dually. The only (obvious) thing is that this is stronger than just simplicity. If you find the homework boring, it is your problem for extra credit.