Propositional Proof Complexity

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Homework 2

1. Let $d = \sqrt{n \log s}$, $a = \left(1 - \frac{d}{2n}\right)$, C is a fixed constant and assume that the function $T(n, f)$ is defined by the recursion

$$T(n, 0) \stackrel{\text{def}}{=} n^{Cd};$$
$$T(n, f) \stackrel{\text{def}}{=} 2n \cdot T(n-1, \lceil af \rceil) + T(n-1, f).$$

   Prove that $T(n, f) \leq \exp(O(\sqrt{n \log s}))$.

2. Give a direct (that is, without referring to Hilbert's Nullstellensatz) proof of the Boolean Nullstellensatz: a system $\{ f_i(x_1, \ldots, x_n) = 0 \mid i \in [m] \}$ of polynomial equations does *not* have 0-1 solutions if and only if there exist polynomials P_i, Q_j such that $\sum_i f_i P_i + \sum_j Q_j (x_j^2 - x_j) = 1$.

3. Let k be a field of characteristic 17, and $d_{PC}^k(\tau)$ be the minimal degree of a Polynomial Calculus refutation of τ over k.

   (a) Prove that $s_R(PHP_n^m)$ and $s_R(FPHP_n^m)$ are[1] non-increasing in m.

   (b) Prove that $d_{PC}^k(PHP_n^m)$, $d_{PC}^k(FPHP_n^m)$ and $d_{PC}^k(onto{-}PHP_n^m)$ are non-increasing in m.

   (c) Calculate $d_{PC}^k(onto - FPHP_n^{n+12})$.

4. Prove the completeness result for Cutting Planes.

[1] I do not know if this is also true for $onto - PHP_n^m$

5. Let G be an undirected graph, and $\tau_{VC}(G)$ be the affine constraints $\{ x_u + x_v \geq 1 \mid (u, v) \in E(G) \}$ defining its vertex-cover polytope. Prove that for every triangle (u, v, w), the *odd-cycle* constraint $x_u + x_v + x_w \geq 2$ has a rank 1 derivation from $\tau_{VC}(G)$ in:

   (a) Cutting Planes (rank 1 = one application of the division rule);

   (b) *LS*.

   How about cycles of length 17?