Kernel-Size Lower Bounds: The Evidence from Complexity Theory

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IAS

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Part 1/3
Note

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Main works discussed

Part 1: introduction to the OR- and AND-conjectures and their use. Covers [BDFH’07], [DvM’10], [DM’12].

Part 2: Evidence for the OR-conjecture [FS’08]

Part 3: Evidence for the AND-conjecture (and OR-conjecture for probabilistic reductions) [D’12]
• **P vs. NP**: The central mystery of TCS.

• Can’t understand this problem, but would like to use

  \[ P \neq NP \]

  hypothesis to “explain” why many tasks are difficult.
These talks: describe how (an extension of) $P \neq NP$ can explain hardness of kernelization tasks.

Our focus: building the initial bridge between these two domains.

[Many other papers]: clever reductions between kernelization problems, to show dozens of kernel lower bounds (LBs).
1. Introduction

2. OR/AND-conjectures and their use

3. Evidence for the conjectures
Introduction
Problems and parameters

Input: Formula $\psi$.

Is $\psi$ satisfiable?

Parameters of interest:

- total bitlength;
- $\#$ clauses;
- $\#$ variables;
- can invent many more measures.
Problems and parameters

Our view in these talks:

- computational problems can have multiple interesting parameters.
- won’t define parameters formally, but always will be easily measurable.

\[ x \rightarrow k(x) \]

- Insist: \( k(x) \leq |x| \)
A parametrized problem $P$ with associated parameter $k$ is Fixed-Parameter Tractable (FPT) if some algorithm solves $P$ in time

$$f(k(x)) \cdot \text{poly}(|x|).$$
Self-reduction for problem $P$: a mapping $R$ s.t.

$x$ a “Yes”-instance of $P \iff R(x)$ a “Yes”-instance of $P$

Goal: want $R(x)$ to be smaller than $x$.

This talk: only interested in poly-time self-reductions.

(Will also discuss reductions between param’d problems...)
Let $F$ be a function.

Poly-time self-reduction $R$ is an $F(k)$-kernelization for $P$ w.r.t. parameter $k$, if:

$$\forall x : |R(x)| \leq F(k(x))$$

Output ("kernel") size bounded by function of the parameter alone!
Virtues of kernels

- $F(k)$-kernels for any (decidable) problem yields an FPT algorithm.
- Many natural FPT algorithms have this form.
Virtues of kernels

- If $F(k) \leq \text{poly}(k)$ and problem is in $\text{NP}$, we get an FPT alg. with runtime
  \[
  \text{poly}(|x|) + \exp(\text{poly}(k)).
  \]
  (compress the instance) (solve reduced instance)

- $F(k) \leq \text{poly}(k)$: “Polynomial kernelization”
Virtues of kernels

- Kernelization lets us *compress* instances to store for the future.

- Also allows us to *succinctly describe* instances to a second, more powerful computer.
Virtues of kernels

- Many great kernelization algs; won’t survey here...

- Which problems fail to have small kernels?
For decidable problems:

- $F(k)$-kernels implies FPT, so...
- NOT FPT implies no $F(k)$-kernels for any $F$!

E.g., $k$-Clique is $W[1]$-complete, so is not FPT or $F(k)$-kernelizable, unless

$$\text{FPT} = W[1]$$
Leaves possibility that all “natural” problems in FPT have poly$(k)$-kernels!
Kernelization limits

- A few kernel size LBs based on $P \neq NP$...

- “Dual parameter” technique [Chen, Fernau, Kanj, Xia ’05] shows that $k$-Planar Vertex Cover has no $1.332k$-kernels* unless $P = NP$.

  * (only applies to reductions that don’t increase $k$)
A few kernel size LBs based on $P \neq NP$...

Similar results for kernels of restricted form, based on NP-hardness of approximation [Guo, Niedermeier ’07].

These bounds are all $\Theta(k)$. 
Lower bound tools were limited, until a paper of [Bodlaender, Downey, Fellows, Hermelin ’07].

Introduced “OR-” and “AND-conjectures,” showed that these would rule out poly($k$)-kernels for many problems.

Related, independent work in crypto: [Harnik, Naor ’06]
Many follow-up works showed the usefulness, versatility of the OR-conjecture for kernel LBs.

We’ll describe one important example:
[Dell, Van Melkebeek ’10]
(and follow-up by [Dell, Marx ’12])
[Fortnow, Santhanam '08] and [D. '12] showed the OR, AND-conjectures follow from a “standard” assumption in complexity, namely

\[ \text{NP} \not\subseteq \text{coNP/poly} \].

(We’ll discuss this assumption...
We now have strong kernel-size LBs for most problems that resisted kernels. E.g.: unless $\text{NP} \subset \text{coNP/poly}$:

1. $k$-Path does not have $\text{poly}(k)$-kernels;

2. Same for $k$-Treewidth;

3. $N$-Clique (param. $N = \#$ vertices), which has a trivial $N^2$ kernel, does not have kernels of size $N^{2-\varepsilon}$.

(For $d$-uniform hypergraphs, we have the tight threshold $N^d$.)
Before telling this story...

What's the real significance of these negative results?
Possible criticisms

“Kernelizations are assumed to be deterministic. That’s too limited.”

Agreed.

- In practice, almost all kernelizations we know are deterministic. But for meaningful lower bounds, we need to understand randomized ones as well.

- But—since [D.’12], our kernel LBs also apply to randomized algorithms.
Possible criticisms

“Kernelizations are assumed to map problem instances to instances of the same problem. That’s also too limited.”

- But all known kernel LBs for NP problems are insensitive to the target problem.
- They apply to “cross-kernelization” as well.
Possible criticisms

“There are applications of kernelization could be achieved under a broader definition. You’re just ruling out one path to those goals.”

Agreed.

- In particular, self-reductions which output many smaller instances (whose solutions yield a solution to the original instance) could be nearly as useful for FPT algs. [Guo, Fellows]

- We don’t understand full power of these “Turing kernels” (yet!)

- Question explored by [Hermelin, Kratsch, Soltys, Wahlstrom, Wu ’10].
Possible criticisms

- Kernelization also useful to succinctly transmit hard problems to a powerful helper. ⇒ Natural to allow 2-way interaction.

- [Dell, Van Melkebeek ’10]: boost our kernel LBs to communication LBs. (More general!)

- OPEN: extend to probabilistic communication.
“Ultimately, kernelization is just one approach to fast algorithms. Many of the LBs are for problems which already have good FPT algs.”

…but this criticism also applies to kernel upper-bound research! Many papers give kernels where good FPT results were already known.
Kernelization is a natural, rich algorithmic paradigm. It’s worthwhile and interesting to understand its strengths and limitations.
Outline

1. Introduction

2. OR/AND-conjectures and their use
[Bodlaender, Downey, Fellows, Hermelin ’07] got this project rolling.

What core idea lies behind their work?

“Many param’d problems can express an OR of a large number of subproblems, without a blowup in the parameter.

“Those problems should resist small kernels... for a shared reason.”
Let $L \subseteq \{0, 1\}^*$. Define the problem OR(L) by:

- **Input:** a list $\langle x^1, \ldots, x^t \rangle$ of binary strings.

- **Decide:** is some $x^j \in L$?

- **Parameter:** $k := \max_j |x^j|$.

To ease discussion, let OR$_=\ (L)$ be the special case where we require that

$$|x^i| = |x^j| = k \quad \forall i, j .$$

(even this special case resists small kernels.)
Define the problem $\text{AND} \subseteq (L)$ by:

- **Input**: a list $\langle x^1, \ldots, x^t \rangle$ of binary strings.
- **Decide**: is every $x^j \in L$?
- **Parameter**: $k := |x^i| = |x^j|, \quad j = 1, 2, \ldots, t$. 

Note: Asking about individual satisfiability of the $\psi_j$s... (even this special case resists small kernels.)
One approach to kernelization: sparsification

May try to identify instances that are “logically least-likely” to lie in \( L \), remove them.
One approach to kernelization: sparsification

May try to identify instances that are “logically least-likely” to lie in $L$, remove them.
Reasonable idea, but no nontrivial kernel size bounds known for $\text{OR}_=(L)$, $\text{AND}_=(L)$...

No equivalence between the two tasks either!
Let $L$ be any NP-complete language.

- **OR-conjecture**: The problem $\text{OR}_=(L)$ does not have $\text{poly}(k)$-kernels.

- **AND-conjecture**: The problem $\text{AND}_=(L)$ does not either.

Slightly “massaged” forms of the conjectures.

[BDFH’07]: equivalent to respective conjectures for $\text{OR}(\text{SAT}), \text{AND}(\text{SAT})$ (but we won’t need this)
[BDFH’07]: The OR-conjecture $\implies$ none of these problems have $\text{poly}(k)$-kernels:

- $k$-Path, $k$-Cycle, $k$-Exact Cycle and $k$-Short Cheap Tour,
- $k$-Graph Minor Order Test and $k$-Bounded Treewidth Subgraph Test,
- $k$-Planar Graph Subgraph Test and $k$-Planar Graph Induced Subgraph Test,
- $(k, \sigma)$-Short Nondeterministic Turing Machine Computation,
- $w$-Independent Set, $w$-Clique and $w$-Dominating Set.

Dozens more in later works.
Consequences

[BDFH’07]: The AND-conjecture \( \implies \) none of these problems have \( \text{poly}(k) \)-kernels:

- \( k \)-Cutwidth, \( k \)-Modified Cutwidth, and \( k \)-Search Number,
- \( k \)-Pathwidth, \( k \)-Treewidth, and \( k \)-Branchwidth,
- \( k \)-Gate Matrix Layout and \( k \)-Front Size,
- \( w \)-3-Coloring and \( w \)-3-Domatic Number,
Connections

OR, AND-conjectures connect to specific parametrized problems through various technical lemmas and reductions.

Here we explain one of the simplest such connections.¹ Still quite powerful.

¹Related to definitions in [Harnik-Naor '06], [BDFH'07], [Bodlaender, Jansen, Kratsch '11]
Claim

Let $L$ be $\text{NP}$-complete, $P$ a parametrized problem, and suppose there is a poly-time reduction $R$ from an instance $\overline{x}$ to $\text{OR}_{\leq}(L)$ to an equivalent instance of $P$, with

$$k(R(\overline{x})) \leq \text{poly}(k(\overline{x})).$$

Then, if $P$ has some $\text{poly}(k)$-kernelization $A$, so does $\text{OR}_{\leq}(L)$ (and the OR-conjecture fails).

Proof.

To kernelize an instance $\overline{x}$ of $\text{OR}_{\leq}(L)$:

$$\overline{x} \rightarrow R(\overline{x}) \rightarrow A(R(\overline{x})) \rightarrow \text{(reduce to } L).$$
Claim

Let $L$ be \textit{NP}-complete, $P$ a parametrized problem, and suppose there is a poly-time reduction $R$ from an instance $\overline{x}$ to $\text{AND}_\leq(L)$ to an equivalent instance of $P$, with

$$k(R(\overline{x})) \leq \text{poly}(k(\overline{x})).$$

Then, if $P$ has some \textit{poly}(k)-kernelization $A$, so does $\text{AND}_\leq(L)$ (and the AND-conjecture fails).

Proof.

To kernelize an instance $\overline{x}$ of $\text{AND}_\leq(L)$:

$$\overline{x} \rightarrow R(\overline{x}) \rightarrow A(R(\overline{x})) \rightarrow \text{(reduce to } L).$$
Using the connections

Let’s see some (easy) examples.

Define $k$-Path:

- **Input:** $\langle G, k \rangle$.
- **Decide:** does $G$ have a simple path of length $k$?

$k$-Path is FPT, runtime $2^{O(k)} \text{poly}(n)$ [Alon, Yuster, Zwick ’95].

But no $\text{poly}(k)$-kernel known.
Using the connections

- Apply our Claim to show that $k$-Path is hard to kernelize.

- How to express an OR of many NP instances of size $s$, as a single instance of $k$-Path, with $k \approx s$?

- Which $L$ to choose?
Using the connections
Using the connections

- Take $L = \text{HAMILTONIAN PATH}$.

- Use graph encoding where length-$k^2$ input encodes a graph on $k$ vertices.

- On the input $\langle G_1, \ldots, G_t \rangle$ to $\text{OR}_= (L)$ (where $|\langle G_i \rangle| = k^2$), output

$$\langle H := \bigcup G_j , k \rangle .$$

- $H$ has a simple $k$-path $\iff$ some $G_i$ has a Ham path.
Using the connections

Define *k*-Treewidth:

- **Input:** \( \langle G, k \rangle \).

- **Decide:** does \( G \) have treewidth* \( \leq k \)?
  
  *(\( tw = \) a monotone measure of graph “fatness”)*

Treewidth an important graph complexity measure for FPT alg. design.

*k*-Treewidth FPT, but treewidth is NP-hard to compute exactly.
Using the connections

- Given graphs $G_1, \ldots, G_t$, if

$$H := \bigcup G_j$$

then

$$\text{tw}(H) := \max_i (\text{tw}(G_i)) ,$$

so

$$\text{tw}(H) \leq k \iff \bigwedge_i [\text{tw}(G_i) \leq k] .$$

- Basis of proof that (AND-Conjecture) $\implies k$-Treewidth has no $\text{poly}(k)$-kernels. We take

$$L := \{ \langle G \rangle : \text{tw}(G) \leq |V(G)|/2 \} .$$
In \textit{k-Treewidth}, \textit{k-Path} examples, choice of \textbf{NP}-complete language $L$ was “obvious,” closely related to param’ed problem.

Time has shown: sometimes “best” choice of $L$ is not obvious, makes reductions easier.

(We’ll see an example...)
We’ve seen a strong framework for ruling out \( \text{poly}(k) \) kernels (modulo the AND-/OR-conjectures).

Shortly after [BDFH'07], Fortnow and Santhanam showed OR-conjecture is true (for deterministic algorithms) if \( \text{NP} \not\subseteq \text{coNP/poly} \). (We’ll come back to this...)

Dell and Van Melkebeek built on [BDFH'07,FS'08] ideas to give tight kernel LBs for problems that do have \( \text{poly}(k) \)-kernels.

How??
First step: studied [FS’08] carefully!

[FS’08] implicitly shows something much stronger than the OR-conjecture.

Important to know this...
Recall $\text{OR}_\leq(L)$: given $x^1, \ldots, x^t$ each of length $k$, compute

$$\bigvee_{i \in [t]} [x^i \in L].$$

Let $t(k)$ be a function, and let $\text{OR}_\leq(L)^{t(\cdot)}$ be the same problem where we further restrict $t = t(k)$.

Focus on “reasonable” $t(\cdot)$: easily computable, and satisfy

$$t(k) \leq \text{poly}(k).$$
Theorem

[FS’08, implicit]: Assume \( \text{NP} \not\subseteq \text{coNP}/\text{poly} \). If \( L \) is \( \text{NP} \)-complete and \( t(k) \leq \text{poly}(k) \), no poly-time reduction \( R \) from \( \text{OR}_\leq(L)^{t(\cdot)} \) to any other problem can achieve output size

\[
|R(\overline{x})| \leq O(t \log t) ,
\]

where \( t = t(k) \), \( k = k(\overline{x}) \).

- E.g., take \( t(k) := k^{100} \).
- Then input \((x^1, \ldots, x^{t(k)}) \) to \( \text{OR}_\leq(L)^{t(\cdot)} \), of size \( k \cdot k^{100} \), cannot be reduced to a kernel of size \(< k^{100} \)!
- Here, \( \text{OR}_\leq(L)^{t(\cdot)} \) trivially has a \( k^{101} \)-kernel, and by [FS’08] it is nearly optimal!
**Corollary:** the \( k \)-Path problem on \( k^{100} \)-vertex graphs does not have kernels of size \( k^{100} \).

So [DvM’10] not really the first ones to prove good fixed-poly kernel LBs. Their achievement:

1. Express \( \text{OR}_\leq(L) \) instances very efficiently within a parametrized problem instance, minimizing parameter blowup;

2. Find a way to “boost” the [FS’08] bound and get truly tight results.
Define the $N$-Clique problem as:

- **Input:** $\langle G, s \rangle$.
  
  ($G$ a graph on $N$ vertices; $s \leq N$)

- **Decide:** Does $G$ have a clique of size $s$?

- **Parameter:** $N$.

- **Natural input size:** up to $N^2$.

Can we compress?
For kernel LBs, goal is to efficiently express an OR of NP instances within an \textit{N-Clique} instance.

Can easily express an OR of NP instances by $\bigcup \ldots$

\textbf{Problem:} blows up the parameter $N$ linearly!
Wasteful, since most potential edges are not used...
Idea: “Pack” many CLIQUE instances into one graph, in a way that creates no large “unwanted” cliques.

Main effort: Find special “host” graph to contain these instances.
Edge-disjoint clique packing: example
Lemma (Packing Lemma for graphs, DvM ’10)

For any $s, t > 0$ there is a graph $G^*$ on

$$s \cdot (s + t^{5+o(1)})$$

vertices. $E(G^*)$ is union of $t$ edge-disjoint cliques $K_1, \ldots, K_t$ of size $s$, and has no other (“unwanted”) $s$-cliques. $G^*$ can be constructed in time $\text{poly}(s + t)$.

- With this lemma we can “embed” $t$ instances of an appropriate problem into $K_1, \ldots, K_t$ respectively.

(Details...)
The packing lemma

**Lemma**

For any $s, t > 0$ there is a graph $G^*$ on

$$s \cdot (s + t^{5+o(1)})$$

vertices. $E(G^*)$ is union of $t$ edge-disjoint cliques $K_1, \ldots, K_t$ of size $s$, and has no other ("unwanted") $s$-cliques.

- Cliques $K_i, K_j$ can intersect in at most one vertex.
- Suggests we consider them as lines in a (finite) plane...
The packing lemma

- Fix any prime $p > s$.
- We’ll build a graph $G$ with $sp$ vertices, and see how large we can take $t$...
- Vertex set: $LP = \mathbb{F}_p \times \{0, 1, \ldots, s - 1\}$. ("left-plane")
The “left-plane”
For each line $\ell$ define

$$K_\ell = \left\{ \left( (x, y), (x', y') \right) : (x, y), (x', y') \in \ell \right\}$$
Lines and line-cliques

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Lines and line-cliques

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Kernel-Size Lower Bounds
The packing lemma

- \( K_\ell, K_{\ell'} \) are edge-disjoint as needed.
  (two points contained in unique line)

- Each \( K_\ell \) is a clique of size \( s! \)
  (so \( sp \) cliques placed in total.)

- **Problem:** many other \( s \)-cliques...
The packing lemma

- **Inspired idea:** restrict the slope of lines we use.

- Choose “special” $A \subset \mathbb{F}_p^*$, and take

  $$E(G^*) = \bigcup_{\text{slope}(\ell) \in A} K_\ell .$$

- $A =$ large set without length-3 arithmetic progressions (3-APs).
The packing lemma

Key claim: The only $s$-cliques in $G^*$ are the $K_\ell$’s we included.

1. Why true?
2. What does it get us?
In our construction, we packed \( p \cdot |A| \) cliques of size \( s \) into \( G^* \).

**Theorem (Salem, Spencer ’42)**

*There is a 3-AP-free set \( A \subset \mathbb{F}_p^* \) with*

\[
|A| \geq p^{1-o(1)},
\]

*constructible in time \( \text{poly}(p) \).*

So to pack \( t \) cliques, we may take \( p \leq s + t^{5+o(1)} \).

Number of verts. \( N = sp \leq s(s + t^{5+o(1)}) \), as needed!
First, no “vertical” edges \(((x, y), (x, y'))\) in \(G^*\).

Thus any \(s\)-clique must have one element from each “column”

\[
\text{col}_j = \mathbb{F}_p \times [j].
\]
If $K_{bad}$ is an $s$-clique in $G^*$ not equal to some $K_\ell$, then there exist some three adjacent columns like so:

\[
\begin{array}{c c c}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

$j \quad j+1 \quad j+2$
Why is key claim true?

Claim: $a_1 + a_3 = 2a_2$
Why is key claim true?

Claim: \(a_1 + a_3 = 2a_2\)

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Why is key claim true?

Claim: $a_1 + a_3 = 2a_2$
Claim: $a_1 + a_3 = 2a_2$
Why is key claim true?

\[ y_2 - y_1 = a_3 \]
Why is key claim true?

\[ y_2 - y_1 = a_3 \]

\[ y_3 - y_2 = a_1 \]
Why is key claim true?

\[ y_2 - y_1 = a_3 \]
\[ y_3 - y_2 = a_1 \]
\[ y_3 - y_1 = 2a_2 \]
Why is key claim true?

Conclude: $a_1 + a_3 = 2a_2$
Why is key claim true?

This contradicts \((A \text{ is 3-AP-free})\). So \(K_{\text{bad}}\) can't exist!
The upshot

**Theorem (DvM’10)**

Fix any \( t(k) \leq \text{poly}(k) \). There is an \( \text{NP} \)-complete language \( L \), and a poly-time reduction from \( \text{OR} \equiv (L)^{t(k)} \) to \( \text{N-Clique} \), whose output instance \( \langle G, s \rangle \) satisfies

\[
N = |V(G)| \leq O(k^2 + k \cdot t^{0.5+o(1)})
\]

Now suppose \( \text{N-Clique} \) had a kernelization \( R' \) with output size bound \( N^{2-\varepsilon} \).

Let \( t(k) := k^C \), for some \( C \gg 1/\varepsilon \). Apply \( R \) to the output of DvM reduction. Maps \( \text{OR} \equiv (L)^{t(k)} \) instance to an \( \text{N-Clique} \) instance of size

\[
O \left( \left[ k^2 + k \cdot k^{0.5C+o(1)} \right]^{2-\varepsilon} \right) = o \left( k^{C-1} \right)
\]
But \( L \) is NP-complete, so [FS’08] tells us: can’t compress \( \text{OR}_{\geq}(L)^{k^C} \) instances even to size \( O(k^C \log k) \)

(of any target problem)...

unless \( \text{NP} \subset \text{coNP/poly} \).

Similar proof: \( N\text{-Clique} \) on \( d \)-hypergraphs does not have \( N^{d-\varepsilon} \)-kernels. Same for \( N\text{-Vertex Cover} \), others.
[Dell, Marx ’12]: simpler proofs of these and related results.

Basic idea: to efficiently compress OR\((L)\) instances into N-Clique, choose L as an NP-complete language with “special structure” (making instances easier to combine in a shared graph)
The “fussy clique” problem

Define $L = \text{FUSSY-CLIQUE}$:

- **Input:** A graph $G$ on $2s^2$ vertices, presented as
  
  $$V(G) = X_1 \cup \ldots \cup X_s \cup Y_1 \cup \ldots \cup Y_s.$$  

- **Require:**
  1. Each $X_i$ ($Y_i$) is an independent set of size $s$;
  2. Each pair $(X_i, X_j)$ is a complete bipartite graph ($i \neq j$). Same for $(Y_i, Y_j)$.

- **Decide:** does $G$ have a clique of size $2s$?

- **NP-complete** ([Dell, Marx ’12], essentially)
Structure of “fussy” graphs

$S = 2$: 

- $X_1$ and $X_2$ connected to each other and to $Y_1$ and $Y_2$.
Structure of "fussy" graphs

$S = 2$:
The “fussy clique” problem

- Easy to compress \( OR=(FUSSY-CLIQUE) \) instance into an \( N\text{-Clique} \) instance.

Given: \( t \) FUSSY-CLIQUE instances

\[
\{ G_{p,q} \}_{p,q \leq \sqrt{t}}.
\]

- Create graph \( G^* \) on \( \sqrt{t} \times 2s^2 \) vertices, in \( s \)-vertex parts

\[
X_{p,i}, Y_{p,i} \quad i \leq s, \ p \leq \sqrt{t}.
\]

- For \( p, q \leq \sqrt{t} \), place copy of \( G_{p,q} \) on vertex-set

\[
\overline{X}_p := X_{p,1}, \ldots, X_{p,s} \cup \overline{Y}_q := Y_{q,1}, \ldots, Y_{q,s}.
\]
The "fussy clique" problem
The “fussy clique” problem
Analysis: First: if some $G_{p,q}$ has an $2s$-clique, so does $G^*$.

Now suppose $G^*$ has an $2s$-clique $C$. Can intersect only one $\overline{X}_p$, and one $\overline{Y}_q$.

Every $G_{p,q'}$ using $\overline{X}_p$ adds the same edges within $\overline{X}_p$. Similarly for $\overline{Y}_q$.

Thus, $C$ must be a clique in $G_{p,q}$!

Have reduced $OR=(FUSSY-CLIQUE)^t$ to an $N$-Clique instance $\langle G^*, 2s \rangle$ with $N = |V(G^*)| \leq O(s^2 \sqrt{t})$.

Here $|\langle G_{p,q} \rangle| \approx s^4$, so reduction is good enough to infer the same kernel-size lower bounds we got from [DvM’10].