Kernel-Size Lower Bounds: The Evidence from Complexity Theory

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Part 2/3
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Outline

1. Introduction
2. OR/AND-conjectures and their use
3. Evidence for the conjectures
OR/AND-conjectures and their use
Evidence for the OR, AND conjectures:

**Theorem**

Assume \( \text{NP} \not\subset \text{coNP/poly} \). If \( L \) is \( \text{NP} \)-complete, \( t(k) \leq \text{poly}(k) \),

1. **[Fortnow-Santhanam’08]** No deterministic poly-time reduction \( R \) from \( \text{OR}_= (L)^{t(\cdot)} \) to any problem can have output size
   \[
   |R(x)| \leq O(t \log t) .
   \]

2. **[D.’12]** No probabilistic poly-time reduction \( R \) from
   \[
   \text{OR}_= (L)^{t(\cdot)}, \text{AND}_= (L)^{t(\cdot)}
   \]
   to any problem, with \( \Pr[\text{success}] \geq .99 \), can achieve
   \[
   |R(x)| \leq t .
   \]
Let’s back up and discuss:

What does \( NP \not\subseteq \text{coNP/poly} \) mean?

Why believe it?
We use ordinary model of Boolean circuits: $\land, \lor, \neg$ gates, bounded fanin.

Say that decision problem $L$ has poly-size circuits, and write $L \in \text{P/poly}$, if

\[
\exists \{ C_n : \{0,1\}^n \rightarrow \{0,1\} \}_{n>0} : \\
\text{size}(C_n) \leq \text{poly}(n), \quad C_n(x) \equiv L(x).
\]

Non-uniform complexity class: def’n of $C_n$ may depend uncomputably on $n$!

Example: if $L \subseteq 1^*$, then $L \in \text{P/poly}$. Also, $\text{BPP} \subseteq \text{P/poly}$.
Recall: decision problem $L$ is in $\text{NP}$ if:

\[ \exists \text{ poly-time algorithm } A(x, y) \text{ on } n + \text{poly}(n) \text{ input bits} : \]

\[ x \in L \iff \exists y : A(x, y) = 1. \]
Say that decision problem $L$ is in $\text{NP/poly}$ if:

$$\exists \text{ poly-sized ckts } \{C_n(x, y)\}_n \text{ on } n + \text{poly}(n) \text{ input bits} :$$

$$x \in L_n \iff \exists y : C_n(x, y) = 1 .$$

“Non-uniform $\text{NP}$”
Recall that $\text{coNP} = \{ L : \overline{L} \in \text{NP} \}$.

Complete problem: $\text{UNSAT} = \{ \psi : \psi \text{ is unsatisfiable} \}$.

$\text{coNP/poly} = \{ L : \overline{L} \in \text{NP/poly} \}$. 
Connect questions about non-uniform computation to uniform questions?

- Yes!

Need a broader view of nondeterminism…
Given a circuit \( C(y^1, y^2, \ldots, y^k) \) with \( k \) input blocks, consider a 2-player game where Player 1 wants \( C \rightarrow 1 \), P0 wants \( C \rightarrow 0 \).

Take turns setting \( y^1, \ldots, y^k \).
Games and computation

INPUT: \( x \)

0

1
Games and computation

INPUT: $X$

$y_1$
Games and computation

INPUT: $X$

$0 \quad \rightarrow \quad y_1 \quad \rightarrow \quad y_2 \quad \rightarrow \quad 1$
Games and computation

INPUT: X

y1  y2  y3
Games and computation

INPUT: $X$

$0 \rightarrow y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow 1$
Define \( d \)-ROUND GAME (\( \exists \)) as:

- **Input:** a \( d \)-block circuit \( C(y^1, \ldots, y^d) \).

- **Decide:** on 2-player game where P1 goes first, can P1 force a win? \( (C = 1) \)

Define complexity class \( \Sigma^p_d \)

as set of languages poly-time (Karp)-reducible to \( d \)-ROUND GAME (\( \exists \)).

“\( d \)th level of Polynomial Hierarchy”
Games and computation

- **Facts:** $\text{NP} = \Sigma_1^P$ ("solitaire"); $\Sigma_d^p \subseteq \Sigma_{d+1}^p$.

- **Common conjecture:** for all $d > 0$, $\Sigma_d^p \neq \Sigma_{d+1}^p$.

- Otherwise we could efficiently reduce a $(d + 1)$-round game to an equivalent $d$-round one, and

  how the heck do you do that??
Games and computation

- Games allow us to connect uniform and non-uniform complexity questions:

**Theorem (Karp-Lipton ’82)**

*Suppose* $\text{NP}$ *is in* $\text{P/poly}$.  
*Then, for all* $d > 2$,  

$$\Sigma^p_d = \Sigma^p_2.$$
Games allow us to connect uniform and non-uniform complexity questions:

**Theorem (Yap ’83)**

Suppose $\text{NP} \in \text{coNP/poly}$.
Then, for all $d > 3$,

$$\Sigma^p_d = \Sigma^p_3.$$
So: the assumption

\[ \text{NP} \not\subseteq \text{coNP/poly} \]

can be based on an (easy-to-state, likely) assumption:

“One cannot efficiently reduce a 100-round game to an equivalent 3-round game!”
The minimax theorem

- An extremely useful tool.
- Many applications in complexity theory, beginning with [Yao’77].
- Gives alternate (but similar) proof of [Fortnow-Santhanam’08] result;
- Seems crucial for best results in [D.’12].
The minimax theorem

- **Setting:** a 2-player, simultaneous-move, zero-sum game.

- Players 1, 2 have finite sets $X, Y$. ("possible moves")

- "Payoff function" $\text{Val}(x, y) : X \times Y \rightarrow [0, 1]$.

- $\text{Val}(x, y)$ defines "payoff from P1 to P2," given moves $(x, y)$.

- (P1 trying to minimize $\text{Val}(x, y)$, P2 trying to maximize)
The minimax theorem

- **Mixed strategy for P1**: A distribution $D_X$ over $X$.

- (Mixed strategies can be useful...)

- Minimax thm says: for P1 to do well against all P2 strategies...
  it’s enough if P1 can do well against any fixed mixed strategy.
The minimax theorem

**Theorem (Minimax—Von Neumann)**

Suppose that for every mixed strategy $D_Y$ for P2, there is a P1 move $x \in X$ such that

$$\mathbb{E}_{y \sim D_Y} \left[ \text{Val}(x, y) \right] \leq \alpha .$$

Then, there is a P1 mixed strategy $D_X^*$ such that, for all P2 moves $y$,

$$\mathbb{E}_{x \sim D_X^*} \left[ \text{Val}(x, y) \right] \leq \alpha .$$

- Follows from LP duality theorem.
Back to business

- Time to apply these tools.
- Let’s restate the [Fortnow-Santhanam’08] result.
- Will switch from $k$’s to $n$’s...
Theorem (FS’08, restated)

Let \( L \) be an \( \text{NP} \)-complete language, \( L' \) another language, and \( t(n) \leq \text{poly}(n) \).

Suppose there is a poly-time reduction

\[
R(\overline{x}) = R(x^1, \ldots, x^{t(n)})
\]

taking \( t(n) \) inputs of length \( n \), and producing output such that

\[
R(\overline{x}) \in L' \iff \bigvee_j [x^j \in L].
\]

Suppose too we have the output-size bound

\[
|R(\overline{x})| \leq O(t(n) \log t(n)).
\]

Then, \( \text{NP} \subseteq \text{coNP/poly} \).
To ease discussion:

- Assume \( L' = L \);
- Fix \( t(n) = n^{10} \);
- Assume \[ |R(x^1, \ldots, x^{n^{10}})| \equiv n^3. \]

(No more ideas needed for general case!)
• Recall $L$ is NP-complete. To prove theorem, enough to show that
  
  \[ L \in \text{coNP/poly}, \quad \text{i.e.,} \quad \overline{L} \in \text{NP/poly}. \]

• Thus, want to use $R$ to build a non-uniform proof system witnessing membership in $\overline{L}$. 
For $x \in \{0, 1\}^n$, say that $\overline{x} = (x^1, \ldots, x^{n^{10}})$ contains $x$ if $x$ occurs as one of the $x^j$'s.

Define the shadow of $x \in \{0, 1\}^n$ by

$$\text{shadow}(x) := \{ z = R(\overline{x}) : \overline{x} \text{ contains } x \} \subseteq \{0, 1\}^{n^3}.$$
Shadows

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Shadows

$X^1$  $X^2$  $X$  $X^4$  $X^5$  $X^6$

$Z$
Shadows

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Kernel-Size Lower Bounds
Fact: if some $z \notin L$ is in the shadow of $x$, then $x \notin L$.
(by OR-property of $R$...)

This is our basic form of evidence for membership in $\overline{L}$!
($z$ will be non-uniform advice...)
The main claim

Claim (FS ’08)

There exists a set $Z \subseteq \overline{L}_n^3$, with

$$|Z| \leq \text{poly}(n),$$

such that for every $x \in \overline{L}_n$,

$$\text{shadow}(x) \cap Z \neq \emptyset.$$

Intuition: the massive compression by $R \Rightarrow$ some $z$ is the image of many sequences $\overline{x}$, hence is in many shadows. Can collect these “popular” $z$’s to hit all shadows (of $\overline{L}_n$).

Claim easily implies $\overline{L} \in \text{NP/poly}$… take $Z$ as non-uniform advice.
Shadows

To prove $x \in \overline{L}$...
Shadows

To prove $x \in \overline{L}$... nondeterministically choose $\overline{x} \supset x$ and $z \in Z$, and check:

$X^1 \quad X^2 \quad X \quad X^4 \quad X^5 \quad X^6$
To prove $x \in \overline{L}$... nondeterministically choose $\overline{x} \supset x$ and $z \in Z$, and check:

Conclusion: $x \in \overline{L}$. 
The main claim

Claim (FS ’08)

There exists a set $Z \subseteq \overline{L}_n^3$, with

$$|Z| \leq \text{poly}(n),$$

such that for every $x \in \overline{L}_n$ ,

$$\text{shadow}(x) \cap Z \neq \emptyset.$$
Proof by game

To prove Claim, consider the following simul-move game between P1 ("Maker") and P2 ("Breaker"):

Game

- **P1**: chooses $z \in \overline{L}_{n^3}$;
- **P2**: chooses $x \in \overline{L}_n$;
- **Payoff to P2**: $Val(x, z) = 1$ if $z \notin shadow(x)$, otherwise 0.

Lemma

There is a P1 strategy (dist’n $D^*$ over $\overline{L}_{n^3}$) such that for any $x$,

$$\mathbb{E}_{z \sim D^*}[Val(z, x)] \leq o(1).$$

Our Claim follows easily, with $|Z| = O(n)$. (Repeated sampling!)
There is a P1 strategy \( D^* \) over \( \overline{L}_{n^3} \) such that for any \( x \),

\[
\mathbb{E}_{z \sim D^*}[\text{Val}(z, x)] \leq o(1).
\]

By minimax theorem, it’s enough to beat any fixed P2 mixed strategy

\[
D_n \quad (\text{dist’}n \text{ over } \overline{L}_n).
\]

**Idea:** use P1 strategy induced by outputs of \( R \) on inputs from \( D_n \).
Proof by game

- Say that $z \in \overline{L}_{n^3}$ is bad, if
  \[ \Pr_{x \sim D_n} [z \in \text{shadow}(x)] \leq 1 - 1/n . \]

- We’ve beaten strategy $D_n$ if some $z$ is not bad.
Proof by game

- Let $D_{n}^{\otimes t}$ denote $t$ ind. copies of $D_{n}$.

- Define dist’n $\mathcal{R}$ by

\[
\mathcal{R} = R\left(D_{n}^{\otimes n^{10}}\right)
\]

- We claim that

\[
\Pr_{z \sim \mathcal{R}}[z \text{ is bad }] = o(1).
\]
Proof by game

Let $\overline{x} = (x^1, \ldots, x^{n^{10}}) \sim D_n^\otimes n^{10}$, and
$$z = R(\overline{x}) .$$

Consider any bad $z$. For $[z = z]$, we must have
$$x^j \in \text{shadow}(z) \quad \forall j ,$$
with happens with probability
$$\leq (1 - 1/n)^{n^{10}} < 2^{-n^9} .$$

Union bound over all bad $z$ completes proof:
$$\Pr [z \text{ is bad}] \leq \frac{2n^3}{2n^9} = o(1) .$$
Summary

- If
  \[ R(\overline{x}) : \{0, 1\}^{n \times n^{10}} \to \{0, 1\}^{n^3} \]
  is a compressive mapping with the “OR-property” for \( L \), then there are \( z \in \overline{L}_{n^3} \) lying in the “shadow” of many \( x \in \overline{L}_n \).

- We collect a small set \( Z \) of these non-uniformly, use it to prove membership in \( \overline{L}_n \).

- Note: nondeterminism still required to verify membership in \( \overline{L}_n \): we have to guess extensions \( x \to \overline{x} \) which map to \( z \)!

- Minimax theorem made our job easier.
Fortnow-Santhanam technique applies to randomized algorithms avoiding false negatives. Also to co-nondeterministic alg’s (observed in [DvM '10]).

[FS '08] also used their tools to rule out “succinct PCPs” for NP...

Left open: evidence again two-sided error OR-compression; any strong evidence against AND-compression.

poly(k)-kernelizability of problems like $k$-Treewidth left open...
Side note: a mystery

- Fortnow-Santhanam prove we cannot efficiently compress \( \text{OR} = (L)^{t(n)} \) instances to size \( O(t(n) \log t(n)) \).

- **Input size is** \( t(n) \cdot n \).

- There is still a big gap here; consequences of compression to size \( O(t(n) \cdot \sqrt{n}) \)? If, e.g., \( L = \text{SAT} \)?

- Same issue with [D’12] bounds...