CMSC 22100/32100: Programming Languages

Homework 3

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1. Prove the substitution lemma (Lemma 2.3) given in Section 2.4 of the
   22pt lecture notes.

2. Complete the proof of preservation (Lemma 2.1, same section), *i.e.*, ex-
   12pt plain the reasoning for all of the cases.

3. 12pt Likewise, complete the proof of progress (Lemma 2.2, same section).

4. The following set of equations defines *size* as a metric on terms for the language in section 2 of the lecture notes:

$$\begin{aligned}
\mathit{size}(n) &= 0 \\
\mathit{size}(x) &= 0 \\
\mathit{size}(e_1 + e_2) &= \mathit{size}(e_1) + \mathit{size}(e_2) + 1 \\
\mathit{size}(e_1 * e_2) &= \mathit{size}(e_1) + \mathit{size}(e_2) + 1 \\
\mathit{size}(\textbf{let } x = e_1 \textbf{ in } e_2) &= \mathit{size}(e_1) + \mathit{size}(e_2) + 1
\end{aligned}$$

   Based on this metric, prove that the language is *terminating*, *i.e.*, that every well-formed, closed expressions evaluates to a value after a finite number of steps.

   Hints:

   - You should argue that the number of steps needed to evaluate e is bounded by $\mathit{size}(e)$.
   - First write down a mathematical statement that captures this intuition in terms of $e \mapsto^1 e'$, $\mathit{size}(e)$, and $\mathit{size}(e')$.
   - Now prove this statement. You will argue by induction. Induction on what?
   - You will need a special version of a substitution lemma that tells you something about the *size* function when it is applied to the result of
     30pt a substitution. State and prove this lemma separately.

5. Which of the following pairs (S, R) of a set S and a binary relation R on S are well-founded? Justify each of your answers:

   (a) S consists of the positive natural numbers $1, 2, 3, \ldots$ and aRb iff a
   8pt divides b and $a \neq b$.

(b) S consists of the rational numbers greater or equal than 1 where aRb
8pt if $a < b$ under their natural ordering.

(c) S consists of finite ASCII strings where aRb iff a is a proper sub-string
8pt of b.