Remarks on the Theory of Two-Player Games*

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In his book "On Numbers and Games", John Conway introduced a notion of equivalence between two-player games. He then showed that the equivalence classes form an ordered abelian group, under an appropriate notion of addition. His construction is reminiscent of the traditional construction of natural numbers from finite sets, or even of the Grothendieck group in an abelian category. However, Conway does not say what it is to be a transformation (a morphism) between two games, and thereby fails to obtain a very attractive category of games. We propose, in the short note that follows, to describe this category, which is really a "combinatorial" calculus of strategies.

Game, player, play, winning, losing. . .

Recall that a *game* G is essentially a "set" equipped with two membership relations

$$h \in^L G \text{ and } g \in^R G$$

$g \in^R G$ means that g is a *position* that the Right *player* could take if he opens the game. Similarly, $h \in^L G$ means that h is a position that the Left player could take if he opens the game.

To complete this description, we must assume that *the opening positions g and h are themselves games*. We add the hypothesis that a game is well-founded: every chain of positions $g \ni x_1 \ni x_2 \cdots$ is finite, without regard for the alternation of occurences of $\ni^R$ or $\ni^L$ in the chain.

If these conditions are satisfied, we define a *play* to be a chain of positions $G \ni x_1 \ni x_2 \cdots \ni x_n$, where the occurences of $\ni^R$ and $\ni^L$ alternate and which, in addition, is maximal in the sense that if the chain ends with $x_{n-1} \ni^L x_n$, then the right player has no opening move in the game x_n (because the set of such positions is empty); the right player is then the *loser*, because the last player to play (in a particular play) is the *winner*. (Those plays that start with $G \ni^R x_1$ are clearly the plays that are opened by the Right player.)

We can write

$$G = (\{h : h \in^L G\}, \{g : g \in^R G\})$$

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to indicate that a game is completely determined by its opening positions (for Left and for Right). This notation allows us to define

1. The empty game $0 = (\emptyset, \emptyset)$, which has no opening positions. The first player immediately loses this game!

2. The opposite $-G$ of a game G,

$$-G = (\{-g : g \in^R G\}, \{-h : h \in^L G\})$$

where we reverse the roles of the players Left and Right.

3. The sum $H + G$ of two games,*

$$\begin{aligned} H + G &= (\{h + G : h \in^L H\} \cup \{H + g : g \in^L G\}, \\ &\qquad \{H + g : g \in^R G\} \cup \{h + G : h \in^R H\}) \end{aligned}$$

Remarks The two previous definitions are inductive definitions: the sum $H+G$ is determined if we know the sums $h + G$, $H + g$, etc. A less formal description of the sum: the two player have both games H and G in front of them; to respond to the oppoent, a player first decides which one of the games H and G to play, then chooses (in that game only) a position which follows legally from the current position of that game. Then it is the turn of the other player to do the same.

Winning Strategy

A winning strategy is a rule that dictates (to the player who uses it during a play) a choice of position with which to respond to the position chosen by the opponent. A player who applies a winning strategy will be the winner, because his opponent could not be the last to play. We write $S_0(G)$ for the set of winning strategies for the Left player in a game G, assuming that the Right player begins the game. An element $U \in S_0(G)$ is therefore a rule which allows us to choose a position $x_{2n-1} \ni^L x_{2n}$ whenever we are given a chain

$$G \ni^R x_1 \ni^L x_2 \ni^R \cdots \ni^R x_{2n-1}$$

in which each position of the form x_{2r} for $1 \leq r < n$ has previously been chosen using the rule U.

We write $S_1(G)$ to represent the set of winning strategies for the Left player, assuming that he (Left) begins the game himself. Then we have

$$S_0(G) \cong \prod_{g \in^R G} S_1(g) \qquad (1)$$

because choosing a strategy $T \in S_0(G)$ is equivalent to choosing a strategy $T_g \in S_1(g)$ for each position $g \in^R G$.

*Translator's note: The original typescript has $(\{h + G : h \in^L H\}, \{H + g : g \in^R G\})$ which seems to be incorrect. I have substituted the definition here.

We also have

$$S_1(G) \cong \sum_{g \in^L G} S_0(g) \tag{2}$$

because choosing a strategy $U \in S_1(G)$ is equivalent to choosing a position $g \in^L G$ and a strategy $h \in S_0(g)$.

The formulae (1) and (2) completely determine $S_0(G)$ and $S_1(G)$ by (transfinite) recursion on the positions of G.

Calculus of Strategies

We shall now describe a category Y whose objects are the two-player games. A morphism

$$G \xrightarrow{f} H$$

will be a winning strategy for the left player in the game $H - G = H + (-G)$, over those plays which the Right player starts. A morphism $f \in Y(G, H)$ is then a strategy $f \in S_0(H - G)$.

1. The *identity* $G \xrightarrow{1_G} G$ is the strategy that consists (for the player who plays second in $G - G$) of, for every move made in one component of $G + (-G)$, playing the dual move in the other component.
2. *Strategy on the sum $G + H$.*
   We can easily define the sum $T + U$ of two strategies $T \in S_0(G)$, $U \in S_0(H)$,
   $$S_0(G) \times S_0(H) \xrightarrow{+} S_0(G + H)$$
3. *Residual strategy.*
   We shall define an operation
   $$S_0(A) \times S_0(B - A) \xrightarrow{*} S_0(B)$$
   This is the main point of this note. It is a matter of describing a strategy $T * U \in S_0(B)$, derived from the pair of strategies $(U, T) \in S_0(A) \times S_0(B - A)$.

   Suppose that the Right player opens a play in B. The Left player could well consider this to be an opening in $B + (-A)$. The strategy T then permits him to respond in $B + (-A)$. There are two cases:

   (a) The strategy T dictates a response in the component B of the game $B + (-A)$. In this case, he plays this move in B and waits for the Right player to play (on B) his chosen next move.

   (b) The strategy T dictates a response in the component $-A$ of the game $B + (-A)$. In this case, the play would be suspended if the strategy $-U$ did not make it possible to simulate a Right move on $-A$. (It is a simulation because the Right player can go on ignoring the existence of the game $B + (-A)$.)

Thus we see that the Right player (real or simulated) replies on $B+(-A)$ to the Left player's strategy T. In case (3a) she uses all her freedom and intelligence; in case (3b) the Left player simulates her response on $-A$, by using $-U$. This induces on B a play which the Right player will lose, because the Left player will win on $B+(-A)$ and lose on $-A$ (since U is winning on A) which means that she will win on B. The Left player need make no effort, she does nothing but use the resources of the strategies T and U. Therefore she has a winning strategy on B: it is the residual strategy $T * U$.

Composition of Strategies

We can now define the composite

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \overset{gf}{\searrow} & \downarrow g \\ & & C \end{array}$$

of strategies $f \in S_0(B+(-A))$ and $g \in S_0(C+(-B))$. First we form the sum

$$g+f \in S_0(C+(-B)+B+(-A)) \cong S_0(C+(-A)+[(-B)+B])$$

We then calculate the residual strategy

$$(g+f) * 1_B \in S_0(C+(-A))$$

Theorem

1. The rule for composing strategies determines a category Y.
2. The sum $G+H$ is a commutative, associative bifunctor with neutral element 0.
3. For every game G, the functor

$$G+(—) : Y \longrightarrow Y$$

is left adjoint to the functor

$$-G+(—) : Y \longrightarrow Y.$$

In summary, the category Y is symmetric monoidal closed and self-dual: just like the category of finite-dimensional vector spaces over a field K. We have the following table of correspondences

0	$\rightarrow$	K
$G+G$	$\rightarrow$	$V \otimes_K W$
$-G$	$\rightarrow$	V^*
$\mathrm{Hom}(G,H)$	$\rightarrow$	$\mathrm{Hom}_K(V,W)$

and the identities

$$\begin{array}{rcl} \mathrm{Hom}(G,H) \cong H - G & \to & \mathrm{Hom}_K(V,W) \cong W \otimes_K V^* \\ \mathrm{Hom}(G+H,L) \cong (\mathrm{Hom}(G,\mathrm{Hom}(H,L)) & \to & \mathrm{Hom}_K(V,\mathrm{Hom}_K(W,Z)) \end{array}$$

etc.

We can now proceed like Conway: two games H and G are equivalent if there exist arrows $H \xrightarrow{f} G$ and $G \xrightarrow{g} H$. This is clearly an equivalence relation. The equivalence classes of games form an abelian group with the addition

$$\begin{array}{lcl} [H] + [G] & = & [H+G] \\ [H] + [0] & = & [H] \\ [H] + [-H] & = & [0] \end{array}$$

It is even an *ordered* abelian group, because we can define an order relation: $[H] \leq [G]$ if there exists an arrow $H \xrightarrow{f} G$. This is the abelian group of surreal numbers. . .