1 Cartesian bicategories

We now expand the scope of application of system Beta, by generalizing the notion of (discrete) cartesian bicategory[1]. Here we give just a fragment of the theory, with a more rounded account to appear elsewhere.

Antecedents of this notion date back as far as Peirce, who noticed an analogy between the algebra of relations and linear algebra. For instance, relational composition

$$RS(a, c) = \exists_{b \in B} R(a, b) \wedge S(b, c)$$

is analogous to matrix composition

$$AB_{ik} = \sum_{j \in J} A_{ij} B_{jk},$$

and relational opposite is analogous to matrix transpose. This type of analogy was developed further by Lawvere [**?**], who observed that relational calculus can be viewed as applying not just to "sets" and "relations", but to objects like categories and modules (what Lawvere calls bimodules; also called profunctors) between them, with module tensor product playing the role of relational composition.

A possible setting which would embrace both examples (sets and relations; categories and modules) and many others, in other words a categorical setting for generalized relational calculus, is the notion of *cartesian bicategory* proposed by Carboni & Walters. This was spelled out precisely for the case of locally ordered bicategories (where the relevant "relations" are partially ordered, and instances of the ordering $R \leq S$ are the 2-cells of the bicategory), which covers for instance bicategories of relations in regular categories. A key feature of their theory is the interplay between a cartesian bicategory $\mathbf{B}$ and the subbicategory $\mathbf{Map}(\mathbf{B})$ whose arrows are the left adjoints in $\mathbf{B}$, called *maps*. In the case of sets and relations, maps are precisely functions, whereas for categories and modules, maps are functors (up to Morita-equivalence for categories).

While there is no question that Carboni & Walters envisioned a notion which would stretch beyond the locally ordered case, and their definition can be read as a template for how the generalization might go, there are technical complications in carrying out the generalization according to their description. For one, they define a cartesian bicategory as a certain type of symmetric monoidal bicategory, a notion which (beyond the case of local orders) didn't exist at the time of their writing, and which by necessity is a complicated notion. (See [**?**] for the "semi-strict" version of this notion.) Then, each object of a cartesian bicategory is to be a cocommutative pseudocomonoid and each arrow a lax morphism of such: it remains to spell out these notions with all their attendant coherence conditions, and pretty soon the notion of cartesian bicategory appears forbiddingly technical and hard to digest.

[1] As far as the authors are aware, the notion of cartesian bicategory in full bicategorical generality has never before been published.

We offer here a fairly simple definition of cartesian bicategory which should ameliorate these technicalities. The basic idea is that a cartesian bicategory is a bicategory $\mathbf{B}$ with just enough "lax structure" so that $\mathbf{Map}(\mathbf{B})$ acquires (2-)products. Using that, we recover the structures adumbrated in the Carboni-Walters style of definition, and show how the lax structure is essentially uniquely determined through its interplay with $\mathbf{Map}(\mathbf{B})$.

1.1 Preliminaries

Our bicategorical terminology is standard (homomorphisms, strong or pseudo transformations, modifications), with a few possible exceptions given below.

A *lax transformation* $\theta : F \to G$ between homomorphisms $F, G : \mathbf{B} \to \mathbf{C}$ assigns 1-cells $\theta b : Fb \to Gb$ to objects of $\mathbf{B}$, and 2-cells θf of the form

$$\begin{tikzcd} Fa \arrow[r, "\theta a"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\overset{\theta f}{\Rightarrow}"] & Ga \arrow[d, "Gf"] \\ Fb \arrow[r, "\theta b"'] & Gb \end{tikzcd}$$

to 1-cells $f : a \to b$ of $\mathbf{B}$, satisfying well-known 2-naturality and coherence conditions. Homomorphisms, lax transformations, and modifications from $\mathbf{B}$ to $\mathbf{C}$ form a bicategory which we denote by $\mathbf{Lax}(\mathbf{B}, \mathbf{C})$.

A lax transformation θ is *strong* if θf is invertible for each 1-cell f. Homomorphisms, strong transformations, and modifications from $\mathbf{B}$ to $\mathbf{C}$ form a bicategory which we denote by $\mathbf{Hom}(\mathbf{B}, \mathbf{C})$.

1.1.1 Bicategorical adjunctions

A *biadjunction* $F \dashv G$ consists of homomorphisms $F : \mathbf{B} \to \mathbf{C}$, $G : \mathbf{C} \to \mathbf{B}$ together with an adjoint equivalence

$$\mathbf{C}(F-, -) \simeq \mathbf{B}(-, G-)$$

in the bicategory $\mathbf{Hom}(\mathbf{B}^{op} \times \mathbf{C}, \mathbf{Cat})$. In elementary terms, a biadjunction consists of homomorphisms F, G as above, strong transformations $\eta : 1_{\mathbf{B}} \to GF$ and $\varepsilon : FG \to 1_{\mathbf{C}}$ (the *unit* and *counit*), and invertible modifications

$$\begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] \arrow[r, phantom, shift right=3ex, "\overset{s}{\Rightarrow}"] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] \arrow[r, phantom, shift right=3ex, "\overset{t}{\Leftarrow}"] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd}$$

called *triangulators* [?], such that the following *triangulator coherence conditions* hold:

$$
\begin{tikzcd}
1 \arrow[r, "\eta"] \arrow[d, "\eta"'] \arrow[dr, phantom, "\overset{\eta\eta}{\Rightarrow}"] & GF \arrow[d, "GF\eta"] \arrow[ddr, "1"] \arrow[dr, phantom, "\overset{Gt}{\Rightarrow}"] & \\
GF \arrow[r, "\eta GF"] \arrow[ddrr, "1"'] & GFGF \arrow[dr, "G\varepsilon F"] \arrow[d, phantom, "\overset{sF}{\Rightarrow}"] & \\
& {} & GF
\end{tikzcd}
= 1_{1\cdot\eta} \quad ; \quad
\begin{tikzcd}
FG \arrow[dr, "F\eta G"] \arrow[drr, "1"] \arrow[ddr, "1"'] & & \\
& FGFG \arrow[r, "\varepsilon FG"'] \arrow[d, "FG\varepsilon"] \arrow[ur, phantom, "\overset{tG}{\Rightarrow}"] \arrow[dl, phantom, "\overset{Fs}{\Rightarrow}"] \arrow[dr, phantom, "\overset{\varepsilon\varepsilon}{\Rightarrow}"] & FG \arrow[d, "\varepsilon"] \\
& FG \arrow[r, "\varepsilon"'] & 1
\end{tikzcd}
= 1_{\varepsilon\cdot 1}.
$$

A *lax adjunction* is defined the same way as a biadjunction except $\eta : 1_{\mathbf{B}} \to GF$ and $\varepsilon : FG \to 1_{\mathbf{C}}$ need not be strong, only lax, and the triangulators s, t need not be invertible. In that case, for any choice of objects b in $\mathbf{B}$ and c in $\mathbf{C}$, there is a local adjunction between hom-categories

$$(L : \mathbf{B}(b, Gc) \to \mathbf{C}(Fb, c)) \dashv (R : \mathbf{C}(Fb, c) \to \mathbf{B}(b, Gc))$$

where $L(g : b \to Gc) = (Fb \xrightarrow{Fg} FGc \xrightarrow{\varepsilon c} c)$ and $R(f : Fb \to c) = (b \xrightarrow{\eta b} GFb \xrightarrow{Gf} Gc)$. The unit of $L \dashv R$ at $g : b \to Gc$ is the pasting

$$
\begin{tikzcd}
b \arrow[r, "g"] \arrow[d, "\eta b"'] \arrow[dr, phantom, "\overset{\eta g}{\Leftarrow}"] & Gc \arrow[d, "\eta Gc"'] \arrow[dr, "1_{Gc}"] \arrow[dr, phantom, "\overset{sc}{\Leftarrow}"] & \\
GFb \arrow[r, "GFg"'] & GFGc \arrow[r, "G\varepsilon c"'] & Gc
\end{tikzcd}
$$

and the counit at $f : Fb \to c$ is

$$
\begin{tikzcd}
Fb \arrow[r, "F\eta b"] \arrow[dr, "1_{Fb}"'] \arrow[dr, phantom, "\overset{tc}{\Leftarrow}"] & FGFb \arrow[r, "FGf"] \arrow[d, "\varepsilon Fb"] \arrow[dr, phantom, "\overset{\varepsilon f}{\Leftarrow}"] & FGc \arrow[d, "\varepsilon c"] \\
& Fb \arrow[r, "f"'] & c.
\end{tikzcd}
$$

The triangular equations for $L \dashv R$ follow from the triangulator coherence conditions. (Warning: it is not generally true that a lax adjunction induces an adjoint pair in $\mathbf{Lax}(\mathbf{B}^{op} \times \mathbf{C}, \mathbf{Cat})$; cf. Lemma 1.3.1.)

1.1.2 Maps

Following Carboni & Walters, we call a left adjoint in a bicategory $\mathbf{B}$ a *map* of $\mathbf{B}$. $\mathbf{Map}(\mathbf{B})$ denotes the locally full subbicategory whose 1-cells are the maps of $\mathbf{B}$. Every homomorphism $F : \mathbf{B} \to \mathbf{C}$ induces a homomorphism $F| : \mathbf{Map}(\mathbf{B}) \to \mathbf{Map}(\mathbf{C})$ by restriction, and every lax transformation $\theta : F \to G$ restricts to a *strong* transformation in $\mathbf{Hom}(\mathbf{Map}(\mathbf{B}), \mathbf{C})$, by the following proposition.

Proposition 1.1.1 *If $f : b \to c$ is a map of $\mathbf{B}$, then θf is invertible.*

Proof: Let f^* denote a right adjoint of f. It is easily verified that $(\theta f)^{-1}$ is given by the pasting

$$\begin{tikzcd}
Fb \arrow[r, "Ff"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3, "\Rightarrow"] & Fc \arrow[r, "\theta c"] \arrow[d, "Ff^*"] \arrow[dr, phantom, "\overset{\theta f^*}{\Rightarrow}"] & Gc \arrow[d, "Gf^*"] \arrow[dr, "1"] \arrow[dr, phantom, shift right=3, "\Rightarrow"] & \\
& Fb \arrow[r, "\theta b"'] & Gb \arrow[r, "Gf"'] & Gc
\end{tikzcd}$$

where the triangles are induced by the unit and counit of $f \dashv f^*$. **QED**

A lax transformation θ is *map-valued* if θb is a map for each object b of the domain bicategory. By Proposition 1.1.1, a map-valued lax transformation $\theta : F \to G$ in $\mathbf{Lax}(\mathbf{B}, \mathbf{C})$ restricts to a strong transformation $\theta| : F| \to G|$ in $\mathbf{Hom}(\mathbf{Map}(\mathbf{B}), \mathbf{Map}(\mathbf{C}))$.

1.2 Definition of cartesian bicategory

Let $\mathbf{B}$ be a bicategory. In what follows, $\Delta : \mathbf{B} \to \mathbf{B} \times \mathbf{B}$ denotes the diagonal homomorphism, $\mathbf{1}$ denotes the terminal bicategory, and $! : \mathbf{B} \to \mathbf{1}$ denotes the unique homomorphism.

Definition 1.2.1 *A* cartesian structure *on* $\mathbf{B}$ *consists of the following data:*

- *Homomorphisms* $\otimes : \mathbf{B} \times \mathbf{B} \to \mathbf{B}$, $I : \mathbf{1} \to \mathbf{B}$,
- *Map-valued lax transformations* $\delta : 1_{\mathbf{B}} \to \otimes\Delta$, $\pi : \Delta\otimes \to 1_{\mathbf{B}\times\mathbf{B}}$, $\varepsilon : 1_{\mathbf{B}} \to I!$,
- *Invertible modifications*

$$\begin{tikzcd}
\otimes \arrow[r, "\delta\otimes"] \arrow[dr, "1_\otimes"'] \arrow[dr, phantom, shift left=3, "\overset{s}{\Rightarrow}"] & \otimes\Delta\otimes \arrow[d, "\otimes\pi"] \\
& \otimes
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta \arrow[r, "\Delta\delta"] \arrow[dr, "1_\Delta"'] \arrow[dr, phantom, shift left=3, "\overset{t}{\Leftarrow}"] & \Delta\otimes\Delta \arrow[d, "\pi\Delta"] \\
& \Delta
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\varepsilon I"] \arrow[dr, "1_I"'] \arrow[dr, phantom, shift left=3, "\overset{u}{\Rightarrow}"] & I!I \arrow[d, "I \cdot id"] \\
& I
\end{tikzcd}$$

satisfying triangulator coherence conditions, so that biadjunctions of the form

$$(\Delta : \mathbf{Map}(\mathbf{B}) \to \mathbf{Map}(\mathbf{B}) \times \mathbf{Map}(\mathbf{B})) \dashv (\otimes : \mathbf{Map}(\mathbf{B}) \times \mathbf{Map}(\mathbf{B}) \to \mathbf{Map}(\mathbf{B}))$$

$$(! : \mathbf{Map}(\mathbf{B}) \to \mathbf{1}) \dashv (I : \mathbf{1} \to \mathbf{Map}(\mathbf{B}))$$

are induced by restriction (using the remark after Proposition 1.1.1).

These biadjunctions say that $\otimes$ restricts to a *2-product* on $\mathbf{Map}(\mathbf{B})$, and I is *2-terminal* in $\mathbf{Map}(\mathbf{B})$[2]. Such bicategorical limits are essentially uniquely determined, i.e. uniquely determined up to equivalences which in turn are unique

[2] We use the term '2-product' to refer to a *bicategorical* limit, avoiding 'biproduct' since this can mean an object which is simultaneously a product and coproduct. To be consistent, we also use the terms '2-terminal' and '2-universal'.

up to unique isomorphism. In **??** we show that this essential uniqueness extends to the global structures on $\mathbf{B}$ in Definition 1.2.1, so a bicategory $\mathbf{B}$ admits essentially just one cartesian structure, if any. A *cartesian bicategory* is a bicategory which admits a cartesian structure.

Examples:

1. Consider the bicategory $\mathbf{Mod}$ whose objects are small categories, whose arrows $R : A \to B$ are functors $B^{op} \times A \to \mathbf{Set}$ (left A- right B-modules), and whose 2-cells are natural transformations between such functors. Composition is module tensor product, as suggested by the co-end formula
$$(SR)(c, a) = \int^{b:B} S(c, b) \times R(b, a).$$
Identities $1_A : A \to A$ in $\mathbf{Mod}$ are hom-functors $A(-,-) : A^{op} \times A \to \mathbf{Set}$, according to the Yoneda lemma. This bicategory $\mathbf{Mod}$ is cartesian.

   For objects A, B in $\mathbf{Mod}$, $A \otimes B$ is the categorical cartesian product; for arrows, one has the formula
$$(R \otimes S)(\langle b, d\rangle, \langle a, c\rangle) = R(b, a) \times S(d, c)$$
where $R : A \to B$ and $S : C \to D$ are modules.

   The bicategory of maps in $\mathbf{Mod}$ is biequivalent to the 2-category of small cauchy-complete categories and functors between them (a category is *cauchy-complete* if every idempotent morphism splits; cf. [**?**]). For, the category of modules from A to B is equivalent to the category of modules from A to $\bar{B}$, the cauchy- (or idempotent splitting-) completion of B, so B and $\bar{B}$ are equivalent in $\mathbf{Mod}$ by Yoneda; every functor $f : A \to B$ gives rise to a left adjoint module $B(-, f-) : B^{op} \times A \to \mathbf{Set}$ with right adjoint $B(f-, -)$; every left adjoint module into $\bar{B}$ arises in this way.

2. Other cartesian bicategories with special properties are $\mathbf{Mod}_{\mathrm{gpd}}$, the bicategory of modules between small groupoids, and $\mathbf{Mod}_{\mathrm{ord}}$, the bicategory of modules between partially ordered sets (viewed as categories). As categories, partially ordered groupoids are equivalent to sets, and modules between them form a bicategory better known as $\mathbf{Span}$.

3. These examples generalize in various directions. There is a cartesian bicategory of modules internal to any topos. In a different direction, working in enriched category theory, there is a cartesian bicategory of V-enriched modules between small V-categories whenever V is a cocomplete cartesian closed category. Notice that in this context, the notion of V-groupoid and V-order make sense; taking $V = (0 \to 1)$, V-$\mathbf{Mod}_{\mathrm{gpd}}$ is biequivalent to $\mathbf{Rel}$, the bicategory of sets and relations.

1.3 Monoidal bicategory structure

Let **B** be a cartesian bicategory. The finite 2-product structure on **Map**(**B**) gives **Map**(**B**) a symmetric monoidal bicategory structure[3]. We give a simple argument that the symmetric monoidal structure on **Map**(**B**) extends to **B**.

1.3.1 B is symmetric monoidal

The associativity constraint on **B**,

$$\alpha(a, b, c) : (a \otimes b) \otimes c \to a \otimes (b \otimes c),$$

is defined by regarding a, b, c as objects in **Map**(**B**) and using the associativity there. The associativity on **Map**(**B**) has an expression in terms of $\otimes$, δ, π which are globally defined on **B**, so that expression defines a lax transformation α on **B**. By similar reasoning, the symmetry and unit constraints on **Map**(**B**) extend to lax transformations on **B**. We show in a moment that all of these lax constraints are strong equivalences on **B**.

Symmetric monoidal structure on **B** also demands various structural modifications (such as a Yang-Baxter modification $R_{\bullet|\bullet,\bullet}$) satisfying certain coherence conditions, but the components of such modifications ($R_{a|b,c}$ say) are defined by regarding their arguments a, b, c as objects in **Map**(**B**) and using the appropriate modifications there. Again, each such modification on **Map**(**B**) is definable in terms of biadjunction data $\otimes$, I, δ, π, ε, s, t, u which are globally defined on **B**, so each is a modification on **B**. Various coherence conditions on the modifications must be checked, but the conditions hold at every choice of arguments in **B** by regarding the arguments as objects of **Map**(**B**) and using the symmetric monoidal structure there, so the conditions hold on **B**.

We now show that the associativity, symmetry, and unit constraints are strong equivalences on **B**. In **Map**(**B**), the associativity α has a left adjoint (in fact a quasi-inverse) α^- with components

$$\alpha^-(a, b, c) : a \otimes (b \otimes c) \to (a \otimes b) \otimes c,$$

and (like α) α^- extends to a lax transformation in $\mathbf{Lax}(\mathbf{B} \times \mathbf{B} \times \mathbf{B}, \mathbf{B})$. The unit and counit isomorphisms for $\alpha^- \dashv \alpha$, being expressible in terms of biadjunction data which are globally defined on **B**, are modifications in $\mathbf{Lax}(\mathbf{B} \times \mathbf{B} \times \mathbf{B}, \mathbf{B})$. Thus α, and by similar reasoning the symmetry and unit constraints, are strong transformations by the following lemma:

Lemma 1.3.1 *If $\alpha^- \dashv \alpha$ is an adjunction in* $\mathbf{Lax}(\mathbf{C}, \mathbf{B})$*, then α is strong. If $\alpha^- \dashv \alpha$ is an adjoint equivalence (whence $\alpha^- \dashv \alpha$ and $\alpha \dashv \alpha^-$), then α and α^- are strong.*

[3]That finite 2-products are symmetric monoidal is taken as a given. The proof appeals to universality arguments, analogous to the proof that ordinary categorical products are symmetric monoidal. We touch upon these in **??**.

Proof: Given any 1-cell $r : c \to d$ in $\mathbf{C}$, let $ev_c : \mathbf{Lax}(\mathbf{C}, \mathbf{B}) \to \mathbf{B}$ be the homomorphism which evaluates at c, and let $ev_r : ev_c \to ev_d$ be the evident *colax* transformation. If α is a right adjoint in $\mathbf{Lax}(\mathbf{C}, \mathbf{B})$, then by dualizing Proposition 1.1.1, $ev_r(\alpha) = \alpha r$ is invertible, hence α is strong. **QED**

This completes the argument that the symmetric monoidal structure on $\mathbf{Map}(\mathbf{B})$ extends to $\mathbf{B}$.

1.3.2 Strictification

Being a monoidal bicategory, $\mathbf{B}$ may be regarded as a one-object tricategory [**?**]. By coherence for tricategories, $\mathbf{B}$ is triequivalent to a **Gray**-monoid, i.e. a monoid in the symmetric monoidal closed category **Gray** identified in [**?**]. (The objects of **Gray** are 2-categories, the morphisms are 2-functors, and $\mathbf{hom}(\mathbf{B}, \mathbf{C})$ for 2-categories $\mathbf{B}$, $\mathbf{C}$ is the 2-category of 2-functors, strong transformations, and modifications from $\mathbf{B}$ to $\mathbf{C}$: this describes the closed structure.) Henceforth the diagrammatic notation we use will be as if $\mathbf{B}$ were a **Gray**-monoid (despite the fact we will call $\mathbf{B}$ a bicategory, not a 2-category). This notational shift requires that $\mathbf{B} \times \mathbf{B}$ be replaced by the Gray tensor product $\mathbf{B} \otimes_G \mathbf{B}$ (for which there is a standard biequivalence $\mathbf{B} \times \mathbf{B} \simeq \mathbf{B} \otimes_G \mathbf{B}$), and the monoidal product on $\mathbf{B}$ by a 2-functor again denoted by $\otimes$:

$$\otimes : \mathbf{B} \otimes_G \mathbf{B} \to \mathbf{B}$$

(despite the fact we will call $\otimes$ a homomorphism, not a 2-functor). The diagonal on $\mathbf{B}$ is replaced by the composite

$$\mathbf{B} \overset{\Delta}{\to} \mathbf{B} \times \mathbf{B} \simeq \mathbf{B} \otimes_G \mathbf{B}.$$

In particular, in a diagram such as

$$\begin{tikzcd} a \arrow[r, "\delta a"] \arrow[d, "r"'] \arrow[dr, phantom, "\overset{\delta r}{\Rightarrow}"] & a \otimes a \arrow[d, "r \otimes r"] \\ b \arrow[r, "\delta b"'] & b \otimes b \end{tikzcd}$$

$r \otimes r$ is defined to be $(r \otimes 1)(1 \otimes r)$ in $\mathbf{B} \otimes_G \mathbf{B}$, following the general definitional conventions for cubical tricategories ([**?**], p. 39).

We make some further adjustments to bring our notation in conformity with [**?**]: the object I will usually be denoted by 1, and occasionally it is convenient to rewrite the components of $\pi : \Delta \otimes \to 1_{\mathbf{B} \times \mathbf{B}}$, i.e. the projections

$$(\pi_1, \pi_2) : (b \otimes c, b \otimes c) \to (b, c),$$

in terms of ε and the **Gray**-monoid structure, e.g.,

$$(\pi_1 : b \otimes c \to b) = (b \otimes c \overset{b \otimes \varepsilon c}{\to} b \otimes 1 = b).$$