A Topologically-Inspired Approach to Geometric Unstructured Multigrid Methods for Elliptic Problems

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Goals:

- Solve elliptic equations on complex or graded domains
- Optimal unstructured multigrid – difficult
- Topological mesh layout helps us accomplish this

Problem Definitions

- Poisson's Equation: $-\Delta u = f$
- Solve with Finite Elements
- Efficiently solve with finite elements and multigrid on simple domains/meshes

Figure: Simple Mesh Sequence

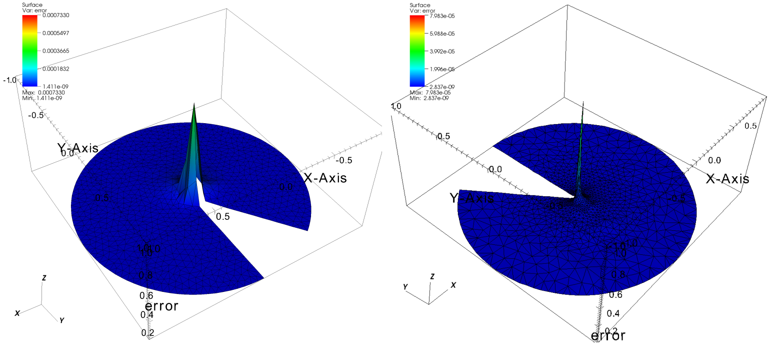
Multigrid

- Relaxation methods: **smooth error persists**
- Solve for smooth error components on coarser meshes
- Formal Definitions:
  - Meshes: $M_0 \rightarrow M_n$
  - The smoother for each M
  - Interpolation operators $R_1 \rightarrow R_n$
- Optimal solve
- Non-nested meshes
  - Cells in M_{fine} potentially span cell boundaries in M_{coarse}.
  - Includes **Node-Nested** Meshes

Error Estimation and Refinement

- Solution properties and **mesh geometry** introduce error
- *A Priori* Refinement
  - Reentrant corner at P_s with angle ω
  - Satisfy: $Ch^{1-\mu} < h < Chr_{P_s}^{1-\mu}$, $\mu > \frac{\pi}{\omega} \in [0, 1]$
    - Start with evenly refined mesh with max. area A_{max}
    - Refine each cell with $r_c = dist(center(c), P_s)$
    - $A_c < r_c^{dim*(1-\mu)} A_{max}$
- *A Posteriori* refinement

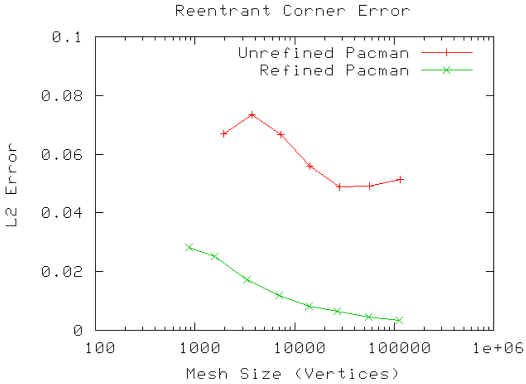
Singularity Error



(a) regular area, 1089 vertices (b) refined singularity, 1044 vertices

Figure: Effects of Local Refinement

Reentrant Mesh Error



Fine Mesh MUST match geometric/solution refinement

Sieve Topology Representation

A **Sieve** represents a Topology (Grothendieck)
Implemented by a Directed Acyclic Graph
Arrows Represent Covering
Operations:

- $Cone(a)$ – The points directly covering a
- $Support(a)$ – The points directly covered by a
- $Meet(a, b)$ – Points Mutually Reachable from points a, b
- $Join(a, b)$ – Points Can Mutually Reach a, b

Finite Element Expression in Sieve

Mesh Components

- A **Sieve** representing the topology of the mesh
- A **Section**, values ⇒ points on the sieve

Labels Sets of sieve points
Depth and Height $depth(0) \Rightarrow$ vertices, $height(0) \Rightarrow$ cells
Coordinates of mesh vertices stored as a section
Associate DOF's ⇒ topological features
works well with element assembly packages (FIAT)

Mesh Coarsening

- Easy in structured case; not so easy unstructured
- Delaunay coarsening – a popular method
  - $M_{fine} \rightarrow$ nonadjacent vertex subset $\rightarrow M_{coarse}$
  - Reduces to a maximal independent set over mesh edges
  - Enforces a spacing increase between vertices
  - Mesh degradation from repeated coarsenings

Function-Based Coarsening

{Miller, Talmor, Teng}

- Vertex **spacing function** – distance to **nearest neighbor**
- Expand the spacing function by C
- Prune the mesh until expanded function is satisfied
    - Remove nodes until spheres of diameter $C * dist_{NN}$ disjoint
- Guaranteed vertex spacing and cell shape
- Works in **any dimension**

Implementation Using Sieve

Vertex spacing through topological traversal of M_{fine}

- M_{fine} topologically reduced forming M_{coarse} vertex subset
- Boundaries preserved through reachability

Interpolation through traversal of M_{fine} and M_{coarse}

- DOFs in M_{fine} connected to topological feature of M_{coarse}
- Generalizable to higher order/exotic elements

Experimental Setup

PETSc (http://www.mcs.anl.gov/petsc/)

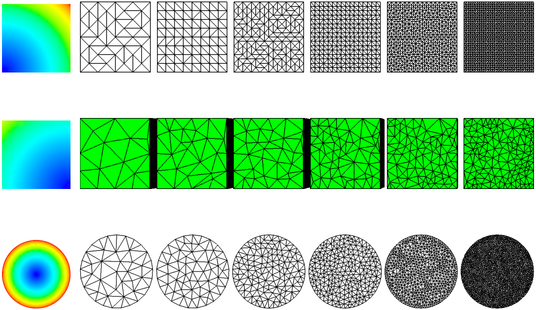
- KSP (Krylov Solvers)
  - DMMG (multigrid framework)

Sieved Meshes
P1 Lagrange Elements tabulated using FIAT
Performance Measures:

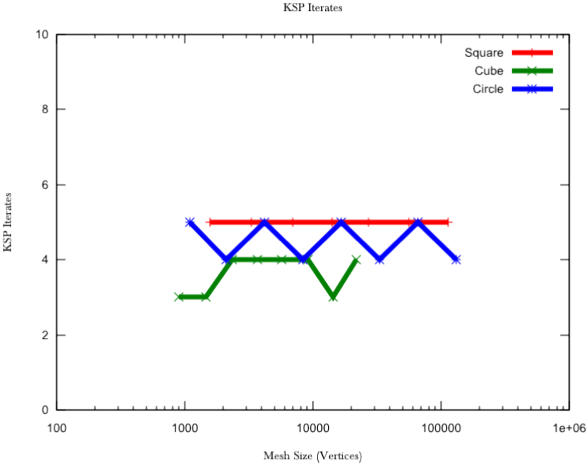
- KSP Iterates
- Point-to-point comparisons during coarsening process

Convex Domains

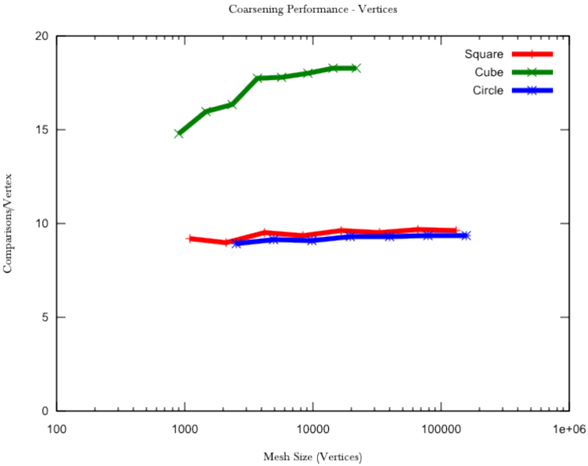
- $\Omega_{square} = [0,1] \times [0,1] (\times [0,1])$
- $\Omega_{circle} = \{p(x,y) : x^2 + y^2 <= 1\}$
- $\Delta u = f$
- $f(x,y) = -4$
- Exact Solution: $u(x,y) = x^2 + y^2$



KSP Performance

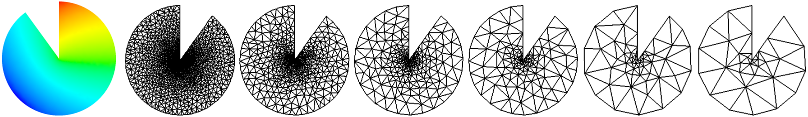


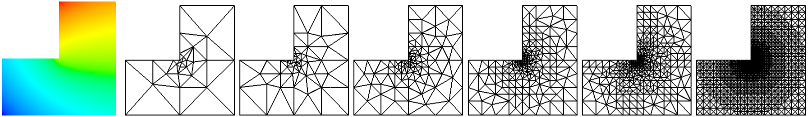
Coarsening Performance



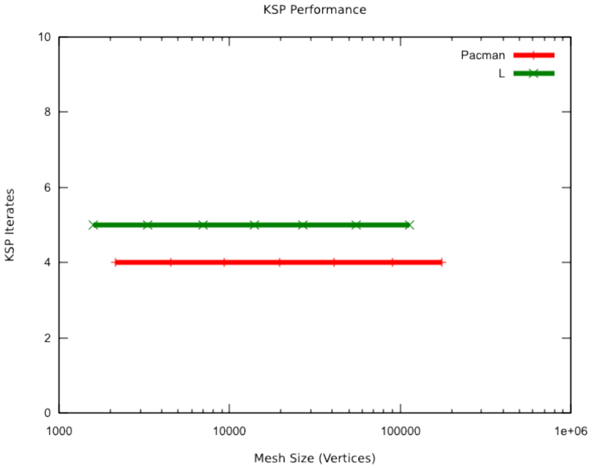
Domains with Reentrant Corners and Refinement

- $\Omega_{pacman} = \{p(x,y) \rightarrow p(r,\theta) : [0,1] \times [0, .9 * 2\pi]\}$
- $\Omega_L = [0,1] \times [-1,1] \setminus [-1,0] \times [-1,0]$
- $\Delta u = f$
- $f(x,y) = 0$
- Exact Solution: $u(x,y) = r^{\frac{2}{3}} sin(\frac{2}{3}\theta)$

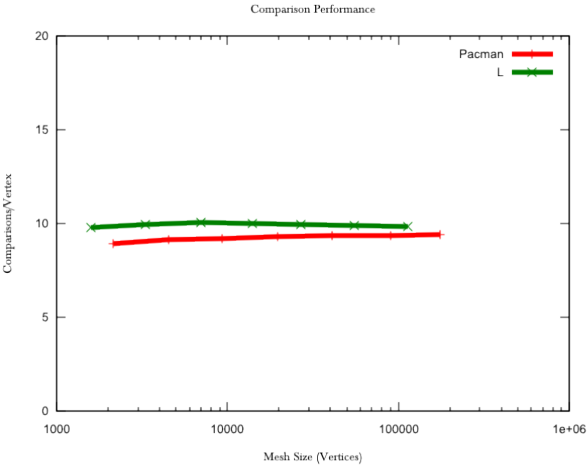




KSP Performance



Coarsening Performance



Summary

- Problems Require Tuned Meshes; Efficient Solutions
- Generalized topology/element FEM representation.
- Arbitrary Finest Mesh → Optimal-Order MG Method

Future Work

- Extend to FAS
- Multiple Coarse Hierarchies for Coupled Problems

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