Dissipation and Spectrum

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References


• P. Constantin, Q. Nie and S. Tanveer, Bounds for second order structure functions and energy spectrum in turbulence Phys. Fluids 11 (1999), 2251-2256.

Navier-Stokes Equations

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \nu \Delta u + f
\]
\[
\nabla \cdot u = 0.
\]

\(f\) = deterministic, time independent. Domain \(D \subseteq \mathbb{R}^3\). No slip boundary conditions conditions.

**Energy Dissipation**

\[
\epsilon = \nu \langle |\nabla u(x, t)|^2 \rangle.
\]

\(\langle \ldots \rangle\) is space-time average.

Long time average

\[
Mu = M_T(u) = \frac{1}{T} \int_0^T u(\cdot, t) dt
\]

Notation:

\[
F^2 = \frac{1}{|D|} \int_D |f|^2 dx
\]

\[
L^{-1} = F^{-1} \|\nabla f\|_{L^\infty(dx)}
\]

\[
U^2 = \limsup_{T \to \infty} \frac{1}{|D|} M_T \int_D |u|^2 dx = \langle |u|^2 \rangle
\]
\[ \epsilon \sim U^3/L \]

Upper bound:
\[ \epsilon \leq \frac{U^3}{L} + \sqrt{\epsilon} \frac{U}{L} \]

This implies, of course,
\[ \epsilon \leq 2 \frac{U^3}{L} + \nu \frac{U^2}{L} \]

and, if one does not like 2, one can use the quadratic equation...
\[ \epsilon \leq \frac{U^3}{L} + \frac{\nu U^2}{4L^2} + \frac{\sqrt{\nu U}}{2L} \sqrt{\frac{\nu U^2}{L^2} + \frac{4U^3}{L}} \]

Proof: (Foias) WLOG \( \nabla \cdot f = 0 \).

\[ f_i = \partial_j M(u_j u_i) + \partial_i M p - \nu \Delta M u + Mu_t \]

because \( M f = f \). Take scalar product with \( f \).
\[
\int_D |f|^2 dx = - \int_D (\partial_j f_i) M(u_j u_i) dx \\
+ \nu \int_D \nabla f \cdot \nabla M(u) dx + \frac{1}{T} (\int_D f \cdot (u(\cdot, T) - u(\cdot, 0)) dx.
\]
It follows that:

$$F \leq \frac{U^2}{L} + \frac{\nu}{F|D|} \int_D |\nabla f| |\nabla Mu| \, dx + O\left(\frac{1}{T}\right)$$

But

$$|\nabla Mu|^2 \leq M |\nabla u|^2$$

(very sad...), so anyway,

$$F \leq \frac{U^2}{L} + \sqrt{\nu} \sqrt{\epsilon/L} + O\left(\frac{1}{T}\right).$$

But

$$\epsilon \leq FU.$$
Littlewood-Paley Decomposition

Nonnegative, nonincreasing, radially symmetric function
\[ \phi_0(k) = \phi(0)(k) \]
\[ \phi_0(k) = 1, \forall k \leq \frac{5}{8}k_0, \]
\[ \phi_0(k) = 0, \forall k \geq \frac{3}{4}k_0. \]

The positive number \( k_0 \) is the wavenumber unit.

\[ \phi_n(k) = \phi(2^{-n}k), \quad \psi_0(k) = \phi(1)(k) - \phi(0)(k) \]
\[ \psi_n(k) = \psi(0)(2^{-n}k), \quad n \in \mathbb{Z}. \]
\[ \psi_n(k) = 1, \forall k \in 2^n[\frac{3}{4}k_0, \frac{5}{4}k_0], \]
\[ \psi_n(k) = 0, \forall k \notin 2^n[\frac{5}{8}k_0, \frac{3}{2}k_0]. \]

1 = \( \phi_n(k) + \sum_{m=n}^{\infty} \psi_m(k), \forall n \in \mathbb{Z} \).

\[ I = S^{(n)} + \sum_{m=n}^{\infty} \Delta_m, \forall n \in \mathbb{Z}. \]
Littlewood-Paley operators $S^{(m)}$ and $\Delta_n$: multiplication, in Fourier representation by $\phi^{(m)}(k)$ and, respectively by $\psi^{(n)}(k)$.

For mean-zero functions $F$ that decay at infinity, $S^{(m)}F \rightarrow 0$ as $m \rightarrow -\infty$.

The LP decomposition is:

$$F = \sum_{n=-\infty}^{\infty} \Delta_n F$$

$$\Delta_n F = \int_{\mathbb{R}^3} \Psi^{(n)}(y)(\delta_y F) dy = F^{(n)}.$$

$$\Psi^{(n)}(y) = (2\pi)^{-3} \int e^{iy\cdot\xi} \psi^{(n)}(\xi) d\xi$$

and

$$(\delta_y F)(x) = F(x-y) - F(x).$$

$\Delta_n$ is a weighted sum of finite difference operators at scale $2^{-n}k_0^{-1}$ in physical space. For each fixed $k > 0$ at most three $\Delta_n$ do not vanish in their Fourier representation at $k$:

$$\Delta_n F(k) \neq 0 \Rightarrow n \in I_k = [-1, 1] + \log_2\left(\frac{k}{k_0}\right).$$
LP Evolution for NSE

\[ u(x, t) = \sum_{n=-\infty}^{\infty} u(n)(x, t) \]

where

\[ u(n)(x, t) = \int_{\mathbb{R}^3} \Psi(n)(y) u(x-y, t) dy = (\Delta_n u)(x, t) \]

Force

\[ f(x) = \sum_{n=-\infty}^{\infty} f(n)(x) \]

Navier-Stokes Littlewood-Paley components equation:

\[ (\partial_t + u \cdot \nabla - \nu \Delta) u(n) + \nabla p(n) = W(n) + f(n) \]

where \( p(n) = \Delta_n p \), are the Littlewood-Paley components of the pressure and

\[ W(n)(x, t) = \int_{\mathbb{R}^3} \Psi(n)(y) \partial_{y_j} (\delta_y(u_j)(x, t) \delta_y(u)(x, t)) dy. \]
2D NS

\[(\partial_t + u \cdot \nabla - \nu \Delta) \omega = f\]

with

\[u(x, t) = \frac{1}{2\pi} \int \frac{y^\perp}{|y|^2} \omega(x - y, t) dy\]

LP vorticity 2DNS

\[(\partial_t + u \cdot \nabla - \nu \Delta) \omega(n) = f(n) + W(n),\]

where

\[W(n)(x, t) =
- \int (\partial_{y_j} (\Psi(n)(y)))(\delta_y u_j)(x, t)(\delta_y \omega)(x, t) dy.\]
Surface QG

\[(\partial_t + u \cdot \nabla + v\Lambda) \theta = f\]

\[u = \nabla^{\perp} \Lambda^{-1} \theta\]

\[\Lambda = (-\Delta)^{1/2}\]

LP SQG

\[\partial_t \theta_{(n)} + u \cdot \nabla \theta_{(n)} + v\Lambda\theta_{(n)} = W_{(n)} + f_{(n)}\]

with

\[W_{(n)} = \int \Psi_{(n)}(y) \nabla_y \cdot (\delta_y(u)\delta_y \theta) dy\]
Littlewood-Paley spectrum

Energy spectrum:

\[ E(k) = \int_{l=k} \langle |\hat{u}(l, t)|^2 \rangle dS(l) \]

\[ \int_0^\infty E(k)dk = \langle |u|^2 \rangle \]

LP Spectrum:

\[ E_{LP}(k) = \frac{1}{k} \sum_{n \in J_k} \langle |\hat{u}(n)(k, t)|^2 \rangle \]

with

\[ J_k = [-2, 2] + \log_2 \left( \frac{k}{k_0} \right) \]
The Littlewood-Paley spectrum is closely related to a shell average of the traditional energy spectrum. If $l$ is a wave number whose magnitude $l$ is comparable to $k$, $\frac{k}{2} \leq l \leq 2k$, then

$$\tilde{u}(l, t) = \sum_{n \in J_k} \tilde{u}(n)(l, t)$$

because $I_l \subset J_k$. It follows that

$$\frac{1}{k} \int_{\frac{k}{2}}^{2k} E(l) dl \leq 5E_{LP}(k).$$

Vice versa, because the functions $\psi(n)$, $n \in J_k$ are non-negative, bounded by 1, and supported in $[\frac{5}{32}k, 6k]$ one has also

$$E_{LP}(k) \leq \frac{5}{k} \int_{\frac{5}{32}k}^{6k} E(l) dl$$
2DNS Spectrum

2D Kraichnan spectrum:

\[ E(k) = C \langle \eta \rangle^{\frac{2}{3}} k^{-3}. \]

\( \eta = \nu |\nabla \omega|^2 \) is the rate of dissipation of enstrophy.

**Theorem 1** Consider forcing with spectrum located in \( k \leq k_f \). For any \( 0 < a < 1 \) such that \( k_f \leq ak_d \) there exists a constant \( C_a \) such that the Littlewood-Paley energy spectrum of solutions of two dimensional forced Navier-Stokes equations obeys the bound

\[ E_{LP}(k) \leq C_a k^{-3} \]

for \( k \in [ak_d, k_d] \). Consequently the traditional spectrum obeys

\[ \frac{1}{k} \int_{\frac{k}{2}}^{2k} E(l) dl \leq \tilde{C}_a k^{-3}. \]
\[ \| \omega(n) \|_{L^2(dx)} \leq C 2^{-n} k_0^{-2} \| \nabla \omega(n) \|_{L^2(dx)} \]

\[ \| \delta_y u \|_{L^\infty(dx)} \leq |y| \| \nabla u \|_{L^\infty(dx)} \]

\[ \| \delta_y \omega \|_{L^2(dx)} \leq |y| \| \nabla \omega \|_{L^2(dx)} \]

\[ \int |y| | \nabla_y \{ \Psi_{(m)} (y) \} | \, dy \leq C \]

\[ \| W_{(n)} (\cdot, t) \|_{L^2(dx)} \leq \Gamma(t) 2^{-n} k_0^{-1} \]

\[ \Gamma(t) = \| \nabla u(\cdot, t) \|_{L^\infty(dx)} \| \nabla \omega(\cdot, t) \|_{L^2(dx)} \]
3D NS Spectrum

Kolmogorov spectrum:

\[ E(k) = C \langle \epsilon \rangle^{2/3} k^{-5/3}. \]
\[ \hat{\epsilon} = \nu \left\langle \left| \nabla u \right|^3 \right\rangle^{2/3} \]
\[ \hat{k}_d = \nu^{-3/4} (\hat{\epsilon})^{1/4}. \]

**Theorem 2** Consider forcing with spectrum located in \( k \leq k_f \). Assume solutions of the three dimensional Navier-Stokes equation satisfy \( \hat{\epsilon} < \infty \). For any \( 0 < a < 1 \) such that \( k_f \leq a \hat{k}_d \) there exists a constant \( C_a \) such that

\[ E_{LP}(k) \leq C_a k^{-5/3} \]

holds for \( k \in [a \hat{k}_d, \hat{k}_d] \). Consequently

\[ \frac{1}{k} \int_{k}^{2k} E(l) dl \leq \tilde{C}_a k^{-5/3}. \]

Nie, Tanveer, C: Taylor microscale Reynolds numbers up to \( R_\lambda = 155 \): inertial range with \( E(k) \sim \hat{C} \hat{\epsilon}^{2/3} k^{-5/3} \) with constant \( \hat{C} \).
SQG Inverse cascade

Swinney experiment:

\[ E(k) \sim k^{-2}, \ k \leq k_f. \]

**Theorem 3** Consider forcing with spectrum located in \( k \geq k_f \). Then

\[ E_{LP}(k) \leq Ck^{-2} \]

holds for solutions of SQG, for all \( k < k_f \). The constant has units of length per time squared and is given by

\[ C = c_0 E_1^1 v^{-3} \langle |f(r, t)|^2 \rangle. \]
Ideas of proof

Basic balance

\[ \nu k \langle |\theta_n(k, t)|^2 \rangle = \langle W_n(r, t)\theta_n(r, t) \rangle. \]

for \( k < k_f, n \in I_k \). The inverse cascade region will be described by wave numbers smaller than the minimal injection wave number \( k_f \). The inverse cascade region corresponds thus, in the Littlewood-Paley decomposition, to indices \( n > -\infty \) that satisfy \( 2^{n+1}k_0 < k_f \). We show now that the right hand side of the equation is bounded above, uniformly for all such \( n > -\infty \).

\[ \langle W_n(r, t)\theta_n(r, t) \rangle = -\int \nabla_y \Psi_n(y) \langle \delta_y(u)(r, t)\delta_y(\theta)(r, t)\theta_n(r, t) \rangle \, dy. \]

The term \( \theta_n \) is bounded pointwise by applying the Fourier inversion formula and Schwartz inequality:

\[ |\theta_n(r, t)| \leq c_\psi 2^n E(t)^{\frac{1}{2}} \]

where \( c_\psi^2 = (2\pi)^{-2} \int |\psi(0)(k)|^2 \, dk \) and

\[ E(t) = \int |\theta(r, t)|^2 \, dr = \int |u(r, t)|^2 \, dr \]

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is the instantaneous total energy. In order to bound the other two terms we note that, from Plancherel, we have

\[ \int |\delta_y \theta(r, t)|^2 dr = (2\pi)^{-2} \int dk \left| e^{-iy \cdot k} - 1 \right|^2 |\tilde{\theta}(k, t)|^2. \]

Using \( |e^{-iy \cdot k} - 1|^2 \leq 4yk \), we deduce

\[ \int |\delta_y \theta(r, t)|^2 dr \leq 4y \eta(t), \]

where

\[ \eta(t) = \int \theta(r, t) \Lambda \theta(r, t) dr \]

The term involving \( \delta_y u \) is bounded using the same argument.

\[ \int |\delta_y u(r, t)|^2 dr \leq 4y \eta(t) \]

So

\[ |\langle \delta_y (u)(r, t) \delta_y (\theta)(r, t) \theta_{(n)}(r, t) \rangle| \leq 4c_\psi 2^n y \langle E(t) \right)^{1/2} \eta(t) \rangle \]

In view of the fact that the functions \( \Psi_{(n)} \) are dilates of a fixed function, we deduce that

\[ |\langle W_{(n)}(r, t) \theta_{(n)}(r, t) \rangle| \leq 2^{n+1} C_\psi \eta E^{1/2}. \]

Here \( E = \sup_t E(t) \) and

\[ C_\psi = 2c_\psi \int y |\nabla_y \Psi_{(0)}(y)| dy = c_0 k_0 \]
is proportional to $k_0$ and depends on the choice of the Littlewood-Paley template $\psi_0$ only through the non-dimensional positive absolute constant $c_0$. The number

$$\eta = \langle \eta(t) \rangle$$

is related to the long time dissipation. It can be bound in terms of the forcing term using the SQG

$$\frac{1}{2} \frac{d}{dt} E(t) + v\eta(t) = \int f(r,t) \theta(r,t) d\mathbf{r}.$$ 

Time average:

$$\eta \leq v^{-2} k_f^{-1} \langle |f(r,t)|^2 \rangle$$

This bound diverges for very large scale forcing, i.e. when $k_f \to 0$. Nevertheless, because of the presence of the coefficient $2^{n+1}$ and the fact that $2^{n+1} k_0 \leq k_f$ in the inverse cascade region, the total bound on the spectrum does not diverge as $k_f \to 0$:

$$|\langle W_{(n)}(r,t) \theta_{(n)}(r,t) \rangle| \leq c_0 v^{-2} E_2^{1/2} \langle |f(r,t)|^2 \rangle.$$
Higher order structure functions

Traditional assumptions of scaling:

\[ \langle |\delta_y u|^m \rangle \sim U^m \left( \frac{|y|}{L} \right)^{\zeta_m} \]

\( UL/\nu \to \infty \). Kolmogorov ’41: \( \zeta_m = \frac{m}{3} \). Landau: Intermittency. Kraichnan: passive scalar anomalous scaling.

Fractional structure functions

For arbitrary \( m \geq 1 \) and \( 0 \leq r \leq 1 \) define fractional structure functions

\[ s_{m;r}(u) = \sup_{x_0} \lim_{T \to \infty} \sup_{T_0} \frac{1}{T} \left( \frac{1}{|B_\rho|} \int \int |\delta_y u(x, t)|^m dx \right)^r dt. \]

Assume: scaling down to the dissipation scale, and spatial ergodicity. Then:

\[ \frac{\zeta_{6m}}{6m} \geq \frac{\zeta_{4m}}{4m} - \frac{1}{m}. \]

Rules out \( \zeta_m \sim m^p \) for \( 0 < p < 1 \).
For the proof:

\[ \Psi = |\delta_y u|^m = |q|^m \]

From PDE

\[ \frac{\nu}{4} \left( 1 - \frac{1}{m} \right) |\nabla \Psi|^2 \leq I + II + III \]

\[ I = \partial_{y_j} ((\delta_y u_j) \Psi^2) , \]

\[ II = 2mq \cdot g|q|^{2m-2} . \]

(with \( g_i = \delta_y f - \partial_{x_i}(\delta_y p) \))

\[ III = - (\partial_t + u \cdot \nabla - \nu \Delta) \Psi^2 \]

Use local Sobolev inequality, keep track of non-local terms and \( m \), evaluate at dissipative cutoff.
Bounds for Bulk Heat Transport

Boussinesq Rayleigh Benard

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \sigma \Delta u + \sigma Ra\hat{e}T,
\]

\[
\nabla \cdot u = 0
\]

\[
\frac{\partial T}{\partial t} + u \cdot \nabla T = \Delta T.
\]

Box of height 1 and lateral side \( L \). \( T = 1 \) at the bottom boundary and \( T = 0 \) at top.

\[
\langle |\nabla T|^2 \rangle = N.
\]

\[
\langle |\nabla u|^2 \rangle = Ra(N - 1).
\]

**Theorem 4** There exists an absolute constant \( C \), independent of Rayleigh number \( Ra \), aspect ratio \( L \) and Prandtl number \( \sigma \) such that

\[
N \leq 1 + C\sqrt{Ra}
\]

holds for all solutions of the Boussinesq equations.

Howard, Busse; Doering, C.
Rotating Infinite Prandtl Number Convection

\[(\partial_t + u \cdot \nabla) T = \Delta T\]
\[-\Delta u - E^{-1}v + p_x = 0\]
\[-\Delta v + E^{-1}u + p_y = 0\]
\[-\Delta w + p_z = RT.\]
\[\nabla \cdot \mathbf{u} = 0\]

B. C.: \(((u, v, w), p, T)\) periodic in \(x\) and \(y\) with period \(L\); \(u, v,\) and \(w\) vanish for \(z = 0, 1, T = 0\) at \(z = 1, T = 1\) at \(z = 0\).

\[N = \langle \|\nabla T\|^2 \rangle\]

**Theorem 5** There exist absolute constants \(c_1, ..., c_4\) so that the Nusselt number for rotating infinite Prandtl-number convection is bounded by

\[N - 1 \leq \min \left\{ c_1 R^{\frac{2}{5}}; \ (c_2 E^2 + c_3 E) R^2; \ c_4 R^{1/3} (E^{-1} + \log_+ R)^{\frac{2}{3}} \right\}. \]

Hallstrom, Putkaradze, Doering, C.
Infinite Prandtl Number Equations

Active scalar equation

\[(\partial_t + u \cdot \nabla) T = \Delta T \quad (1)\]

equations of state:

\[-\Delta u + p_x = 0, \quad (2)\]

together with

\[-\Delta v + p_y = 0 \quad (3)\]

and

\[-\Delta w + p_z = RT. \quad (4)\]

\(R\) represents the Rayleigh number. The velocity is divergence-free

\[u_x + v_y + w_z = 0. \quad (5)\]

The horizontal independent variables \((x, y)\) belong to a basic square \(Q \subset \mathbb{R}^2\) of side \(L\). Sometimes we will drop the distinction between \(x\) and \(y\) and denote both horizontal variables \(x\).
The vertical variable $z$ belongs to the interval $[0, 1]$. The non-negative variable $t$ represents time. The boundary conditions are as follows: all functions $((u, v, w), p, T)$ are periodic in $x$ and $y$ with period $L$; $u$, $v$, and $w$ vanish for $z = 0, 1$, and the temperature obeys $T = 0$ at $z = 1$, $T = 1$ at $z = 0$. We write

$$\|f\|_2^2 = \frac{1}{L^2} \int_0^1 \int_Q |f(x, y, z)|^2 dz dx dy$$

for the (normalized) $L^2$ norm on the whole domain. We denote by $\Delta_D$ the Laplacian with periodic-Dirichlet boundary conditions. We will denote by $\Delta_h$ the Laplacian in the horizontal directions $x$ and $y$. We will use $< \cdots >$ for long time average:

$$\langle f \rangle = \limsup_{t \to \infty} \frac{1}{t} \int_0^t f(s) ds.$$ 

We will denote horizontal averages by an over-bar:

$$\overline{f(\cdot, z)} = \frac{1}{L^2} \int_Q f(x, y, z) dx dy.$$
We will also use the notation for scalar product

\[(f, g) = \frac{1}{L^2} \int_0^1 \int_Q (fg)(x, y, z) \, dx \, dy \, dz.\]

The Nusselt number is

\[N = 1 + \langle (w, T) \rangle.\]  

(6)

One can prove using the equation (1) and the boundary conditions that

\[N = \langle \| \nabla T \|^2 \rangle\]  

(7)

and using the equations of state (2 - 4) that

\[\langle \| \nabla u \|^2 \rangle = R(N - 1).\]  

(8)

This defines a Nusselt number that depends on the choice of initial data; we take the supremum of all these numbers. The system has global smooth solutions for arbitrary smooth initial data. The solutions exist for all time and approach a finite dimensional set of functions. If we think in terms of this dynamical system picture then the Nusselt number represents the maximal long time average distance
from the origin on trajectories. Because all invariant measures can be computed using trajectories the Nusselt number is also the maximal expected dissipation, when one maximizes among all invariant measures.

**Bounding the heat flux**

We take a function $\tau(z)$ that satisfies $\tau(0) = 1$, $\tau(1) = 0$, and write $T = \tau + \theta(x, y, z, t)$. The role of $\tau$ is that of a convenient background; there is no implied smallness of $\theta$, but of course $\theta$ obeys the same homogeneous boundary conditions as the velocity. The equation obeyed by $\theta$ is

$$(\partial_t + u \cdot \nabla - \Delta) \theta = -\tau'' - w\tau'$$

(9)

where we used $\tau' = \frac{d\tau}{dz}$. We are interested in the function $b(z, t)$ defined by

$$b(z, t) = \frac{1}{L^2} \int_Q w(\cdot, z)T(\cdot, z)dx.$$
Its average is related to the Nusselt number:

\[ N - 1 = \langle \int_0^1 b(z)dz \rangle. \]

Note that

\[ T - \overline{T} = \theta - \overline{\theta} \]

Also note that from the boundary conditions and incompressibility

\[ w(z, t) = 0 \]

and therefore

\[ b(z, t) = \frac{1}{L^2} \int_Q w(\cdot, z)\theta(\cdot, z)dx. \]

From the equation (9) it follows that

\[ N = \langle -2 \int_0^1 \tau'(z)b(z)dz - \|\nabla\theta\|^2 \rangle + \int_0^1 (\tau'(z))^2 dz. \]

(10)

Now we are in a position to explain the variational method and some previous results. Consider a choice of the background \( \tau \) that is “admissible” in the sense that

\[ \langle -2 \int_0^1 \tau'(z)b(z)dz - \|\nabla\theta\|^2 \rangle \leq 0 \]
holds for all functions $\theta$. Then of course
\[ N \leq \int_0^1 (\tau'(z))^2 \, dz. \]
The set of admissible backgrounds is not empty, convex and closed in the $H^1$ topology. The background method, as originally applied, is then to seek the admissible background that achieves the minimum $\int_0^1 (\tau'(z))^2 \, dz$. Such an approach would predict $N \leq cR_{\Delta}^2$ for this active scalar, just as in the case of the full Boussinesq system. One can do better. Let us write
\[ b(z, t) = \frac{1}{L^2} \int_Q \int_0^{z_1} w_{zz}(x, z_2, t) \theta(x, z) \, dx \, dz \, dz_2 \, dz_1. \]
(11)

It follows that
\[ |b(z, t)| \leq z^2 (1 + \|\tau\|_{L^\infty}) \|w_{zz}\|_{L^\infty(dz; \, L^1(dx))}. \]
(12)

Now we use two a priori bounds. First, one can prove using (9) and (8) that there exists a positive constant $C_\Delta$ such that
\[ \langle \|\Delta \theta\|^2 \rangle \leq C_\Delta \left\{ RN + \int_0^1 \left[ (\tau''(z))^2 + Rz(\tau'(z))^2 \right] \, dz \right\} \]
(13)
holds. Secondly, one has the basic logarithmic bound
\[ \|w_{zz}\|_{L^\infty} \leq CR(1 + \|\tau\|_{L^\infty})[1 + \log_+(R\|\Delta\theta\|)]^2. \] (14)

Using (14) together with (13) in (12) one deduces from (10)
\[ N \leq \int_0^1 (\tau'(z))^2 dz + CR(1 + \|\tau\|_{L^\infty})^2 \left[ \int_0^1 z^2 |\tau'| dz \right] \]
\[ \left[ 1 + \log_+ \left\{ RN + \int_0^1 \left[ (\tau''(z))^2 + Rz(\tau'(z))^2 \right] dz \right\} \right] \] (15)

Choosing \( \tau \) to be a smooth approximation of \( \tau(z) = \frac{1-z}{\delta} \) for \( 0 \leq z \leq \delta \) and \( \tau = 0 \) for \( z \geq \delta \) and optimizing in \( \delta \) one obtains

**Theorem 6** There exists a constant \( C_0 \) such that the Nusselt number for the infinite Prandtl number equation is bounded by
\[ N \leq N_0(R) \]

where
\[ N_0(R) = 1 + C_0 R^{1/3} (1 + \log_+ R)^2 \]