Final with solution

Analysis I

This is a closed books test. Please show all work. Please solve five out of six. Each problem is worth 20 points.

1. (a) State the closed graph theorem.
   (b) Let \( H \) be a Hilbert space with scalar product \( \langle \cdot ; \cdot \rangle \). Let \( A : H \to H \) be a linear operator satisfying:
   \[
   \langle Af; g \rangle = \langle f; Ag \rangle
   \]
   for all \( f, g \in H \). Prove (the Hellinger-Toeplitz theorem): \( A \) is continuous.
   
   Proof: (a) \( A : E \to F \), \( A \) linear, \( E, F \) Banach. If the graph
   \[
   G_A = \{(e, f) \mid f = Ae\} \subset E \times F
   \]
   is closed, then \( A \) is continuous.
   (b) Let \( f_n \to f \), \( Af_n \to g \). Let \( h \in H \) be arbitrary. Then
   \[
   \langle Af - g; h \rangle = \lim_{n \to \infty} \langle f_n; Ah \rangle - \langle Af_n; h \rangle = 0
   \]

2. (a) State the Radon-Nikodym theorem for Radon measures in \( \mathbb{R}^n \).
   (b) Compute \( D_\mu(\nu) \), the Lebesgue-Besicovitch density, when \( \mu = 1_B dx \) where \( B = B(0,1) \) is the unit ball in \( \mathbb{R}^n \) and \( \nu = dx \) is Lebesgue measure. (That is, \( \mu(A) = |A \cap B| \) if \( |A| \) is the Lebesgue measure of \( A \).) Compute also \( D_\nu(\mu) \). What can you say about \( D_\nu(\mu)(x) \) when \( x \in \partial B \)? What about \( D_\mu(\nu)(x) \) when \( x \in \partial B \)?
   
   Proof. (a) If \( \mu, \nu \) are Radon measures in \( \mathbb{R}^n \) and if \( \nu \) is absolutely continuous with respect to \( \mu \) then
   \[
   \nu(A) = \int_A D_\mu(\nu)(x) d\mu
   \]
holds for all Borel sets $A$ with $D_\mu(\nu)(x) = \lim_{r \to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))}$, if all $\mu(B(x,r)) > 0$, infinity otherwise.

(b) $D_\mu(\nu)(x) = 1$ if $x \in B$, $D_\mu(\nu)(x) = \infty$ if $x \notin B$, $D_\mu(\nu)(x) = 2$ if $x \in \partial B$. $D_\nu(\mu)(x) = 1$ if $x \in B$, $D_\nu(\mu)(x) = 0$ if $x \notin B$ and $D_\nu(\mu) = \frac{1}{2}$ if $x \in \partial B$.

3. (a) Let $f, g \in L^1(\mathbb{R}^n)$. Prove that

$$\hat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

holds for all $\xi \in \mathbb{R}^n$. (Here, of course, $f * g$ is the convolution and $\hat{f}$, the Fourier transform of $f$.)

(b) Let $j \in C_0^\infty(\mathbb{R}^n)$, $j \geq 0$, $\int_{\mathbb{R}^n} j(x)dx = 1$. Let $f \in W^{1,p}(\mathbb{R}^n)$, $1 \leq p < \infty$. Prove that

$$\partial_i(j * f) = j * \partial_i f$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and that

$$\lim_{\epsilon \to 0} f_\epsilon = f$$

holds in $W^{1,p}(\mathbb{R}^n)$. Here $f_\epsilon = j_\epsilon * f$ and $j_\epsilon(x) = \epsilon^{-n} j\left(\frac{x}{\epsilon}\right)$.

Proof. (a) If $f, g \in L^1$ then $f * g \in L^1$ by Fubini and triangle inequality. Then

$$\hat{(f * g)}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \left[ \int_{\mathbb{R}^n} f(x - y)g(y)dy \right] dx$$

We write $e^{-ix \cdot \xi} = e^{-iy \cdot \xi}e^{-(x-y) \cdot \xi}$ and use the fact Lebesgue measure is translation invariant:

$$\int_{\mathbb{R}^n} e^{-iy \cdot \xi}g(y)dy \left[ \int_{\mathbb{R}^n} e^{-i(x-y) \cdot \xi} f(x - y)dx \right] = \hat{g}(\xi)\hat{f}(\xi)$$

In view of Fubini, we are done.

(b) If we prove

$$\partial_i(j * f) = j * \partial_i f$$

then the result follows from the corresponding one for $L^p$. It remains to prove that if $\partial_i(j * f) = j * \partial_i f$. By definition

$$\int_{\mathbb{R}^n} \partial_i(j * f)(x)\phi(x)dx = -\int_{\mathbb{R}^n} (\partial_i\phi)(x)(j * f)(x)dx$$
Using Fubini and one integration by parts we have

\[- \int_{\mathbb{R}^n} (\partial_i \phi)(x)(j \ast f)(x)dx = \int_{\mathbb{R}^n} f(y)dy \int_{\mathbb{R}^n} \phi(x) \partial_x j(x - y)dx \]

Now \(\partial_x j(x - y) = -\partial_y j(x - y)\), so using Fubini again,

\[\int_{\mathbb{R}^n} \partial_i (j \ast f)(x) \phi(x) dx = \int_{\mathbb{R}^n} \phi(x) (j \ast \partial_i f)(x) dx\]

4. (a) Recall the inequality proven in class

\[\left[ \int_{\mathbb{R}^n} |f|^{\frac{n}{p-1}} dx \right]^{\frac{n-1}{n}} \leq C \int_{\mathbb{R}^n} |\nabla f| dx\]

for all \(f \in W^{1,1}(\mathbb{R}^n)\). Let \(1 \leq p < n\). Prove that there exists a constant \(C_{n,p}\) such that

\[\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}\]

holds for all \(f \in W^{1,p}(\mathbb{R}^n)\), where \(p^* = \frac{np}{n-p}\). Hint: Take \(g = f^r\) for an appropriate number \(r\) and use a suitable Hölder inequality.

(b) Give an example of a function \(f \in W^{1,2}(\mathbb{R}^2)\) that is unbounded. (Hint: take a radial function, compactly supported, smooth away from the origin, and equal to an appropriate power of \(|\log r|\) near the origin. Determine the powers that work.)

Proof (a). Without loss of generality \(f \geq 0, f \in C^\infty(\mathbb{R}^n)\).

\[\int_{\mathbb{R}^n} f^{p^*}(x) dx = \int_{\mathbb{R}^n} g^{\frac{n}{p-1}}(x) dx\]

requires \(r = \frac{(n-1)p}{n-p}\). Now

\[|\nabla g| \leq r|\nabla f| f^{r-1}\]

and in view of the inequality in class and the Hölder inequality

\[\left[ \int_{\mathbb{R}^n} f^{p^*}(x) dx \right]^{\frac{n-1}{n}} \leq C \|\nabla f\|_{L^p} \left[ \int_{\mathbb{R}^n} f^{q(r-1)}(x) dx \right]^{\frac{1}{q}}\]

where \(q = \frac{p}{p-1}\). Because \(q(r-1) = p^*\) the inequality follows.
(b) If the smooth, compactly supported, radial function looks like $|\log r|^p$ near the origin, it is square integrable. Its derivative is bounded by a constant multiple of $|\log r|^{p-1}r^{-1}$ near the origin. Its square is integrable iff

$$\int_0^e |\log r|^{2(p-1)}r^{-1}dr < \infty$$

Changing variables $x = |\log r|$, we see that $p < \frac{1}{2}$ implies square-integrability of the gradient.

5. (a) State the Fourier inversion theorem in $L^1(\mathbb{R}^n)$.

(b) Let $0 < \alpha < 1$, and let $C^\alpha(\mathbb{R}^n)$ be the Banach space of Hölder continuous functions of order $\alpha$. Let $T : L^1(\mathbb{R}^n) \rightarrow C^\alpha(\mathbb{R}^n)$ be defined by

$$(Tf)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}(1 + |\xi|)^{-(n+1)}  \hat{f}(\xi) d\xi$$

Prove that $T$ is linear, continuous, one-to-one, but not onto. (Hints: for 1-1, prove that $Tf = 0$ implies $\hat{f} = 0$. For “not onto”, for each fixed $\alpha$, the range of $T$ is actually included in a smaller space, namely in $C^\beta(\mathbb{R}^n)$ with $1 > \beta > \alpha$. You could use the open mapping theorem as well: take a function with compact support and integral equal to zero and rescale it appropriately.)

Proof (a). If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$ then

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) d\xi$$

holds $x$-a.e.

(b) We write

$$Tf(x + h) - Tf(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} (e^{ih\cdot\xi} - 1) (1 + |\xi|)^{-(n+1)}  \hat{f}(\xi) d\xi$$

and use $|e^{ih\cdot\xi} - 1| \leq |h|^\alpha |\xi|^{\alpha}$. We obtain

$$\|Tf\|_{C^\alpha(\mathbb{R}^n)} \leq C\|f\|_{L^1(\mathbb{R}^n)}$$

because $\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$. The fact that $f$ is 1-1 follows by integrating against a family of Gaussians: if $Tf = 0$ then

$$0 = \int_{\mathbb{R}^n} Tf(x)e^{ix\cdot\eta}G(\epsilon x)dx = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-n\epsilon}G\left(\frac{\xi - \eta}{\epsilon}\right) (1 + |\xi|)^{-(n+1)} \hat{f}(\xi) d\xi$$
and letting $\epsilon \to 0$ we obtain $\hat{f}(\eta) = 0$.

6. (a) Recall the inequality proven in class

$$
\int_{B(x,r)} |f(y) - f(z)|^p \, dy \leq Cr^{n+p-1} \int_{B(x,r)} |\nabla f(y)|^p \, |y-z|^{1-n} \, dy
$$

valid for all $1 \leq p < \infty$, $r > 0$, $z \in B(x,r)$ and $f \in C^1(\mathbb{R}^n)$.

Let $p > n$. Prove that

$$
|f(y) - f(z)| \leq C r^{1-\frac{n}{p}} \left[ \int_{B(x,r)} |\nabla f(x)|^p \, dx \right]^\frac{1}{p}
$$

holds for any $f \in W^{1,p}(\mathbb{R}^n)$, $r > 0$, $y, z \in B(x,r)$.

(b) Let $p > n$. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Prove that the map

$$i(f) = f,$$

defined for $f \in C_0^\infty(\Omega)$, extends uniquely to a continuous linear map

$$i : W_0^{1,p}(\Omega) \to C^{1-\frac{n}{p}}(\Omega)$$

where $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

Hint, although we did this in class: compare $f(z)$ to the average on the ball, i.e., consider

$$f(z) - \frac{1}{|B(x,r)|} \int_{B(x,r)} f(w) \, dw$$

Proof (a)

$$
\left| f(z) - \frac{1}{|B(x,r)|} \int_{B(x,r)} f(w) \, dw \right| \leq C r^{-n} \int_{B(x,r)} |f(z) - f(w)| \, dw
$$

Using the inequality proved in class with $p = 1$:

$$r^{-n} \int_{B(x,r)} |f(z) - f(w)| \, dw \leq C \int_{B(x,r)} |\nabla f(w)||z-w|^{1-n} \, dw$$

Now by Hölder

$$
\int_{B(x,r)} |\nabla f(w)||z-w|^{1-n} \, dw
\leq C \left[ \int_{B(x,r)} |\nabla f(w)|^p \, dw \right]^\frac{1}{p} \left[ \int_{B(x,r)} |w-z|^{(1-n)q} \right]^\frac{1}{q}
$$
We have
\[
\left[ \int_{B(x,r)} |w - z|^{(1-n)q} \right]^{\frac{1}{q}} \leq \left[ \int_{B(z,2r)} |w - z|^{(1-n)q} \right]^{\frac{1}{q}} \leq Cr^{1-n/p}
\]

(b) By the previous inequality, taking \( x = \frac{y+z}{2} \) and \( r = |y-z| \), we obtain the inequality
\[
|f(y) - f(z)| \leq C|y-z|^{1-\frac{n}{p}} \|f\|_{W^{1,p}(\mathbb{R}^n)}
\]