Math. 314: Review summary for week 1

This is a brief recap of the first week. The reference is Chapter 10 of big Rudin.

We have defined analyticity: \( f \) is analytic at \( z \) if the limit \( \lim_{\zeta \to z} (\zeta - z)^{-1} (f(\zeta) - f(z)) \) exists. Because of the meaning of the limit, this implies in particular that the partials \( \partial_x f \) and \( \partial_y f \) exist and that \( \overline{\partial} f = 0 \) at \( z \). Here \( \partial_x f = \partial_x u + i \partial_x v \) if \( f = u + iv \), and \( \overline{\partial} f = \frac{1}{2} (\partial_x f + i \partial_y f) \). The equation \( \overline{\partial} f = 0 \) is equivalent to the Cauchy - Riemann system \( u_x = v_y \), \( u_y = -v_x \). I used the notation \( u_x = \partial_x u \). The converse is true. If one has two \( C^1 \) functions \( u \) and \( v \) so that they satisfy the C-R system, then \( f = u + iv \) is analytic. Actually one does not need \( C^1 \). That is the content of the Looman-Menchoff theorem. If the pair \( f = u + iv \) is differentiable at \( z = x + iy \), when we view it as a map from \( \mathbb{R}^2 \) to itself, and if it satisfies C-R, then \( f \) is analytic at \( z \): differentiability says \( |f(x + \xi + i(y + \eta)) - f(x + iy) - (a \xi + b \eta) - i(c \xi + d \eta)| = o(|\xi + i \eta|) \) where the constants \( a, b, c, d \) are \( u_x, u_y, v_x, v_y \). C-R says that \( a = d \), \( b = -c \) so that the linear piece is just \( (a + ic)(\xi + i \eta) = f'(z)(\xi + i \eta) \).

We then pursued series. The radius of convergence of \( \sum_n c_n (z - z_0)^n \) was defined and computed \( \frac{1}{R} = \limsup_{n \to \infty} |c_n|^{\frac{1}{n}} \). (\( R \in [0, \infty] \)). We defined holomorphic functions as functions that have locally a convergent power series representation. Such functions are continuous because the power series converge uniformly on small compact disks. The power series obtained from a convergent power series by differentiating it term by term has the same radius of convergence. It follows that holomorphic functions are analytic, and infinitely many times differentiable. Moreover, the power series is locally uniquely defined, \( c_n = (n!)^{-1} f^{(n)}(z_0) \). We generated many holomorphic functions by the formula \( f(z) = \int_{\Lambda} \frac{1}{\phi(\lambda) - z} \mu(d\lambda) \). Here \( \mu \) is any complex measure on the measurable space \( \Lambda \), \( \phi \) is measurable and \( z \in \Omega \subset \mathbb{C} \) with \( \Omega \) an open set disjoint from the image \( \phi(\Lambda) \). We then defined path integrals \( \int_{\gamma} f(z) dz \) for piece-wise \( C^1 \) paths and continuous \( f \). We worried a bit about reparametrizations, sums and the like. We proved that if \( f \) is continuous, \( \gamma \) closed in an open set \( \Omega \) and if an analytic \( F \) exists so that \( f = F' \) in \( \Omega \) then \( \int_{\gamma} f(z) dz = 0 \). Polynomials are examples of such \( f \). We proved the Cauchy-Goursat theorem for a rectangle: \( R \) a closed rectangle, \( f \) continuous and analytic in a neighborhood of \( R \), then \( \int_{\partial R} f(z) dz = 0 \). The theorem remains true if one has a finite number of points \( z \) not on \( \partial R \) where \( f \) is not known to be analytic, but where \( \lim_{\zeta \to z} (\zeta - z) f(\zeta) = 0 \), or if \( f \) is continuous, and the finitely many points where \( f \) is not known to be analytic are anywhere,
including on the rectangle. We then showed that if $D$ is an open disk, $f$ is continuous in it and analytic except at finitely many points then the integral $\int_{\gamma} f(z)dz = 0$ for any closed path that is piecewise $C^1$. We defined the index of a point with respect to a closed path, $i_{\gamma}(z)$ and showed that it is locally constant off the image of $\gamma$, constant in each connected component, integer valued, and vanishes for the unbounded component. We proved the Cauchy formula in a disk: If $D$ is a disk, $f$ is analytic in it and $\gamma$ is a closed path that is piecewise $C^1$ then for any point $z$ in the disk and not on the curve one has $i_{\gamma}(z)f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta-z} d\zeta$. Next we proved that analytic functions in open sets are holomorphic, using the Cauchy formula.