Singular Integrals

Analysis III

1 Calderon-Zygmund decomposition

Let f be an integrable function $\int_{\mathbb{R}^n} |f| dx < \infty$. We will decompose the function at height $\alpha > 0$, $f = g + b$ with $|g| \le C\alpha$ almost everywhere, and with $\int_{\mathbb{R}^n} |b| dx \le C \|f\|_{L^1(\mathbb{R}^n)}$, and additional properties. We denote by $|A|$ the Lebesgue measure of A.

Theorem 1 *Let $f \in L^1(\mathbb{R}^n)$, and let $\alpha > 0$. There exists a countable collection of cubes with sides parallel to the axes, Q_j with disjoint interiors, such that, for each j,*

$$\alpha < \frac{1}{|Q_j|} \int_{Q_j} |f| dx \le 2^n \alpha$$

Consider $\Omega = \cup_j Q_j$ and $F = \mathbb{R}^n \setminus \Omega$. Then,

$$|\Omega| \le \alpha^{-1} \|f\|_{L^1(\mathbb{R}^n)}.$$

Moreover,

$$|f(x)| \le \alpha$$

holds almost everywhere for $x \in F$. There exists a decomposition

$$f(x) = g(x) + b(x)$$

such that $|g(x)| \le 2^n \alpha$ almost everywhere, moreover, for every $1 \le p \le \infty$,

$$\|g\|_{L^p(\mathbb{R}^n)} \le \alpha^{\frac{p-1}{p}} (1 + 2^{np})^{\frac{1}{p}} \|f\|^{\frac{1}{p}}_{L^1(\mathbb{R}^n)}$$

and $b(x) = 0$ for $x \in F$,

$$\int_{Q_j} b(x) dx = 0$$

for each Q_j, and

$$\|b\|_{L^1(\mathbb{R}^n)} \le (1 + 2^n) \|f\|_{L^1(\mathbb{R}^n)}.$$

Proof. We take cubes Q with sides parallel to the axes. We start by using all such cubes with equal side size large enough so that $\|f\|_{L^1(\mathbb{R}^n)} \leq \alpha|Q|$. We take each such cube and divide it into 2^n daughters, Q' with sides parallel to the axes. We ask whether

$$\frac{1}{|Q'|}\int_{Q'}|f|dx \leq \alpha?$$

If the answer is "yes", we continue subdividing. If the answer is "no", we retain the cube as one of the Q_j-s. Clearly, by construction, the fathers Q of the retained daughters had $\frac{1}{|Q|}\int_Q |f|dx \leq \alpha$ and therefore

$$\alpha < \frac{1}{|Q_j|}\int_{Q_j}|f|dx \leq 2^n\alpha$$

holds because $Q_j \subset Q$ and $|Q| = 2^n|Q_j|$. The interiors of cubes Q_j are mutually disjoint, by induction. We consider the set $\Omega = \cup_j Q_j$. We have

$$|Q_j| < \alpha^{-1}\int_{Q_j}|f|dx$$

and therefore we obtain $|\Omega| \leq \alpha^{-1}\|f\|_{L^1(\mathbb{R}^n)}$ just by summing. The Lebesgue differentiation theorem can be stated as follows:

$$f(x) = \lim_{Q\to x}\frac{1}{|Q|}\int_Q f(y)dy$$

holds x-almost everywhere, where Q is the family of cubes that contain x and $Q \to x$ means that we take their diameters to converge to zero. Note that the cubes are not required to be centered at x, but a simple argument shows that

$$M_{cubes}(f)(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f|dx$$

obeys $M_{cubes}(f)(x) \leq cM(f)(x)$, with c depening on dimension only, and $M(f)$ the usual maximal function. If $x \in F$ is not in the exceptional set of measure zero, then the limit of averages of integrals on cubes exists and equals $f(x)$, and there exists a subsequence of cubes Q' containing x whose diameters converge to zero and which were not retained as Q_j-s, so $|f(x)| \leq \limsup_{Q'\to x}\frac{1}{|Q'|}\int_{Q'}|f|dx \leq \alpha$. We define

$$g(x) = \begin{cases} f(x), & \text{if } x \in F \\ \frac{1}{|Q_j|}\int_{Q_j} f(x)dx, & \text{if } x \in Int(Q_j). \end{cases}$$

This defines g almost everywhere. Also, by construction

$$|g(x)| \leq 2^n \alpha$$

almost everywhere. Now

$$\int_F |g|^p dx \leq \alpha^{p-1} \|f\|_{L^1(\mathbb{R}^n)}$$

and

$$\int_\Omega |g|^p dx \leq \alpha^p 2^{np} |\Omega| \leq \alpha^{p-1} 2^{np} \|f\|_{L^1(\mathbb{R}^n)}.$$

Clearly $b = f - g$ is defined almost everywhere and vanishes on F and satisfies $\int_{Q_j} b(x)dx = 0$. Finally, because $|b| \leq |f| + |g|$, we have

$$\int_\Omega |b| dx \leq \|f\|_{L^1(\mathbb{R}^n)} + 2^n \alpha |\Omega| \leq (1 + 2^n) \|f\|_{L^1(\mathbb{R}^n)}$$

and this concludes the proof.

2 Marcinkiewicz Interpolation Theorem

We recall that a sublinear operator is said to be weak type (p,q) if

$$|\{x \mid |Tf(x)| > \alpha\}| \leq \left(\frac{C\|f\|_{L^p(\mathbb{R}^n)}}{\alpha} \right)^q$$

holds for all $\alpha > 0$ with C independent of α and f. This definition applies only to $q < \infty$. For $q = \infty$ the notion is replaced by the strong notion of boundedness. Note that if $\|Tf\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$ the inequality above follows by Chebyshev's inequality. We recall that if $p_1 < p < p_2$ then

$$L^p(\mathbb{R}^n) \subset L^{p_1}(\mathbb{R}^n) + L^{p_2}(\mathbb{R}^n)$$

Indeed, for any $f \in L^p(\mathbb{R}^n)$ and any $\gamma > 0$ we can write

$$f = f_\gamma + f^\gamma$$

with

$$f_\gamma(x) = \begin{cases} f(x), & \text{if } |f(x)| > \gamma \\ 0 & \text{if } |f(x)| \leq \gamma \end{cases}$$

and

$$f^\gamma(x) = \begin{cases} 0, & \text{if } |f(x)| > \gamma \\ f(x) & \text{if } |f(x)| \le \gamma \end{cases}$$

Then clearly

$$\|f_\gamma\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} \le \gamma^{p_1 - p}\|f\|^p_{L^p(\mathbb{R}^n)}$$

and

$$\|f^\gamma\|^{p_2}_{L^{p_2}(\mathbb{R}^n)} \le \gamma^{p_2 - p}\|f\|^p_{L^p(\mathbb{R}^n)}$$

Theorem 2 *Let $1 < r \le \infty$. Assume that T is subadditive and weak type (1,1) and weak type (r,r). Then for every $1 < p < r$ there exists a constant C_p such that*

$$\|Tf\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)}$$

Proof. We take first $r < \infty$. We take $\alpha > 0$ and denote

$$\lambda(\alpha) = |\{x \mid |Tf(x)| > \alpha\}|$$

We decompose $f = f_\gamma + f^\gamma$ as above but with $\gamma = \alpha$. Then, because $|Tf(x)| > \alpha$ implies at least one one of the inequalities $|Tf_\alpha(x)| > \frac{\alpha}{2}$ or $|Tf^\alpha(x)| > \frac{\alpha}{2}$, we deduce from the assumptions

$$\begin{aligned} \lambda(\alpha) &\le \tfrac{C_1}{\alpha}\int |f_\alpha|dx + C_2^r\alpha^{-r}\int |f^\alpha|^r dx \\ &\le C_1\alpha^{-1}\int_{|f|>\alpha}|f|dx + C_2^r\alpha^{-r}\int_{|f|\le\alpha}|f|^r dx \end{aligned}$$

We multiply by $p\alpha^{p-1}$ and integrate. For the first term we use:

$$\begin{aligned} &\int_0^\infty \alpha^{p-1}\alpha^{-1}\left(\int_{|f|>\alpha}|f|dx\right)d\alpha = \\ &\int_{\mathbb{R}^n}|f(x)|\left(\int_0^{|f(x)|}\alpha^{p-2}d\alpha\right)dx = \tfrac{1}{p-1}\int_{\mathbb{R}^n}|f(x)|^p dx \end{aligned}$$

For the second term we use

$$\begin{aligned} &\int_0^\infty \alpha^{p-1}\alpha^{-r}\left(\int_{|f|\le\alpha}|f(x)|^r dx\right)d\alpha \\ &= \int_{\mathbb{R}^n}|f(x)|^r\left(\int_{|f(x)|}^\infty \alpha^{p-1-r}d\alpha\right)dx \\ &= \tfrac{1}{r-p}\int_{\mathbb{R}^n}|f(x)|^p dx \end{aligned}$$

This concludes the proof for $r < \infty$. If $r = \infty$ we know that Tf^α is essentially bounded, and $|Tf^\alpha(x)| \le C_2\|f^\alpha\|_{L^\infty(\mathbb{R}^n)} \le C_2\alpha$ holds almost everywhere. Therefore the measure of the set where $|Tf(x)| > 2C_2\alpha$ is not larger than the measure of the set where $|Tf_\alpha(x)| > C_2\alpha$ and we continue by estimating the distribution function of Tf as above.

3 Singular Integrals

We consider kernels $K \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$ satisfying the following properties:

There exists a constant B such that

$$|K(x)| \le B|x|^{-n} \tag{1}$$

holds for all $0 \neq x \in \mathbb{R}^n$,

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\, dx \le B \tag{2}$$

holds for any $y \neq 0$, and

$$\int_{r_1<|x|<r_2} K(x)dx = 0 \tag{3}$$

holds for any $0 < r_1 < r_2 < \infty$.

The aim is to prove that convolution integrals with these kernels are bounded in L^p, $1 < p < \infty$. We start by making a few general observations. For $\epsilon > 0$ let

$$(C_\epsilon(K))(x) = \begin{cases} K(x) & \text{if } |x| \ge \epsilon, \\ 0 & \text{if } |x| \le \epsilon \end{cases}$$

let

$$(\tau_\epsilon K)(x) = \epsilon^{-n} K\left(\frac{x}{\epsilon}\right)$$

and for $f \in L^p(\mathbb{R}^n)$

$$(\delta_\epsilon f)(x) = f(\epsilon x).$$

We denote by T_K the convolution operator

$$T_K(f)(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy.$$

This operator is well defined if, for instance, $f \in L^1(\mathbb{R}^n)$ and $K \in L^p(\mathbb{R}^n)$. This latter property is not true in general if all we know about K is (1-3). Part of the theorem will be to make sense of the operators. Let us observe that

Proposition 1 i. *If K has properties (1-3) then $\tau_\epsilon K$ has the same properties with the same constant B, uniformly, for all $\epsilon > 0$.*

ii. *For any* $\epsilon > 0$

$$C_\epsilon(K) = \tau_\epsilon(C_1(\tau_{\frac{1}{\epsilon}}(K)))$$

iii. *If* K *has properties (1-3) then* $C_1(K)$ *has the same properties (1-3), with a constant* $B_1 > 0$ *depending on* B *and dimension of space only.*
iv. *If* $K \in L^p(\mathbb{R}^n)$ *and* $f \in L^1(\mathbb{R}^n)$ *then*

$$\left(\delta_{\frac{1}{\epsilon}} T_K \delta_\epsilon\right) f = T_{\tau_\epsilon K} f$$

v. *If* $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ *is a bounded linear operator then the family* $\delta_{\frac{1}{\epsilon}} T \delta_\epsilon$ *is uniformly bounded, i.e.*

$$\sup_{\epsilon>0} \|\delta_{\frac{1}{\epsilon}} T \delta_\epsilon\|_{\mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))} \leq C$$

vi. *If*

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$$

denotes the Fourier transform, then

$$\mathcal{F}(C_\epsilon(K))(\xi) = \mathcal{F}(C_1(\tau_{\frac{1}{\epsilon}} K))(\epsilon\xi)$$

.

Proof. The proofs of **i**, **ii**, **iv**, **v** and **vi** are direct consequences of the definitions and are left as an exercise. The proof of **iii** uses the following important observation: if K is a function obeying (1) and if $\lambda > 1$, $\rho > 0$ then the integral

$$\int_{\rho\leq|x|\leq\lambda\rho} |K(x)| \leq \omega_n B \log\lambda \tag{4}$$

is bounded uniformly, with ω_n the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, independently of ρ. This is used to bound

$$\int_{|x|>2|y|,|x-y|>1,|x|<1} |K(x-y)|dx \leq \omega_n B \log\frac{3}{2}$$

and

$$\int_{|x|>2|y|,|x-y|<1,|x|>1} |K(x)|dx \leq \omega_n B \log 2$$

It follows that if K obeys (1-3), then $C_1(K)$ obeys the same with $B_1 = (1 + \omega_n \log 3)B$ instead of B.

Lemma 1 *Let K obey (1-3). There exists a constant γ depending on dimension of space only so that*

$$\sup_{\epsilon>0}\sup_{\xi\in\mathbb{R}^n}|\mathcal{F}C_\epsilon K(\xi)|\le\gamma B$$

Proof. Clearly, by points **vi** and **i** in Proposition 1 above, it is enough to prove that

$$|\widehat{K}_1(\xi)|\le\gamma B$$

holds for all ξ, where

$$K_1=C_1(K)$$

and K satisfies (1-3). Now clearly, $K_1\in L^2(\mathbb{R}^n)$ (by Proposition 1 above, $C_\epsilon(K)\in L^2(\mathbb{R}^n)$) and

$$\widehat{K_1}(\xi)=\lim_{R\to\infty}\int_{|x|\le R}e^{-ix\cdot\xi}K_1(x)dx.$$

We fix $\xi\neq 0$ and study separately the integrals for $|x|\le\frac{2\pi}{|\xi|}$ and for larger x.

$$\begin{aligned}|I_1(\xi)|=\left|\int_{|x|\le\frac{2\pi}{|\xi|}}e^{-ix\cdot\xi}K_1(x)dx\right|&=\left|\int_{|x|\le\frac{2\pi}{|\xi|}}\left(e^{-ix\cdot\xi}-1\right)K_1(x)dx\right|\\ &\le\int_{|x|\le\frac{2\pi}{|\xi|}}|x||\xi|B|x|^{-n}dx=2\pi\omega_nB.\end{aligned}$$

We consider now the integrals

$$I_2(\xi)=\int_{\frac{2\pi}{|\xi|}\le|x|\le R}e^{-ix\cdot\xi}K_1(x)dx$$

for $R>\frac{3\pi}{|\xi|}$ and show they are uniformly bounded. In order to do so we choose $\eta=\pi\frac{\xi}{|\xi|^2}$ satisfying $e^{-i\xi\cdot\eta}=-1$. Note that $|\eta|=\frac{\pi}{|\xi|}$. We translate by η:

$$\begin{aligned}I_2(\xi)&=\int_{\frac{2\pi}{|\xi|}\le|x|\le R}e^{-ix\cdot\xi}\left(K_1(x)-K_1(x-\eta)+K_1(x-\eta)\right)dx\\ &=\int_{\frac{2\pi}{|\xi|}\le|x|\le R}e^{-ix\cdot\xi}\left(K_1(x)-K_1(x-\eta)\right)dx+\int_{\frac{2\pi}{|\xi|}\le|x|\le R}e^{-ix\cdot\xi}K_1(x-\eta)dx\end{aligned}$$

The last piece is

$$\begin{aligned}\int_{\frac{2\pi}{|\xi|}\le|x|\le R}e^{-i((x-\eta)\cdot\xi+\eta\cdot\xi)}K_1(x-\eta)dx&=-\int_{\frac{2\pi}{|\xi|}\le|x'+\eta|\le R}e^{-ix'\cdot\xi}K_1(x')dx'\\ &=-I_2(\xi)+E(\xi)\end{aligned}$$

where

$$E(\xi) = \int_A e^{-ix\cdot\xi} K_1(x)dx - \int_B e^{-ix\cdot\xi} K_1(x)dx$$

with

$$A = \left\{ x \mid \frac{2\pi}{|\xi|} \le |x| \le R \right\} \setminus \left\{ x \mid \frac{2\pi}{|\xi|} \le |x+\eta| \le R \right\}$$

and

$$B = \left\{ x \mid \frac{2\pi}{|\xi|} \le \ |x+\eta| \le R \right\} \setminus \left\{ x \mid \frac{2\pi}{|\xi|} \le |x| \le R \right\}$$

We have thus

$$2I_2(\xi) = \int_{\frac{2\pi}{|\xi|} \le |x| \le R} e^{-ix\cdot\xi} \left(K_1(x) - K_1(x-\eta)\right) dx + E(\xi)$$

Now

$$A \subset \left\{ x \mid \frac{2\pi}{|\xi|} \le |x| \le \frac{3\pi}{|\xi|} \right\} \cup \left\{ x \mid R - \frac{\pi}{|\xi|} \le |x| \le R \right\}$$

and

$$B \subset \left\{ x \mid \frac{\pi}{|\xi|} \le |x| \le \frac{2\pi}{|\xi|} \right\} \cup \left\{ x \mid R \le |x| \le R + \frac{\pi}{|\xi|} \right\}$$

because $R > \frac{3\pi}{|\xi|}$. Therefore

$$\int_{A\cup B} |K_1| dx \le c\omega_n B$$

with $c > 1$ an absolute constant follows from (1) via (4). Thus

$$|E(\xi)| \le c\omega_n B.$$

On the other hand, because $|x| \ge \frac{2\pi}{|\xi|} = 2|\eta|$,

$$\int_{\frac{2\pi}{|\xi|} \le |x| \le R} |K_1(x) - K_1(x-\eta)|\, dx \le (1 + \omega_n \log 3)B$$

and thus

$$|I_2(\xi)| \le \frac{1}{2}[c\omega_n B + (1 + \omega_n \log 3)B]$$

concludes the proof of the lemma.

Theorem 3 *Let $K \in L^2(\mathbb{R}^n)$ satisfy (1-3) and*

$$\sup_{\xi \in \mathbb{R}^n} |\widehat{K}(\xi)| \leq \gamma B$$

for some $\gamma > 0$. Then for each $1 < p < \infty$, the operator

$$f \mapsto T_K(f) = \int_{\mathbb{R}^n} K(x-y)f(y)dy$$

defined for $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ has a unique bounded extension

$$T_K : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

and

$$\|T_K\|_{\mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))} \leq C$$

holds with $C = C(n, p, B)$ a uniform constant.

Proof. Because of the uniform bound on the Fourier transform, we have immediately the result for $p = 2$. We will prove that T_K is weak type (1, 1), and conclude from the Marcienkewicz theorem that T_K is bounded in $L^p(\mathbb{R}^n)$ for $1 < p < 2$. Then we will use duality to deduce the theorem for $p > 2$. Let $\alpha > 0$. Let $f \in L^1(\mathbb{R}^n)$ We consider the Calderon-Zygmund decomposition at height α. We have

$$T_K(f) = T_K(g) + T_K(b)$$

Therefore,

$$|\{x||T_K(f)(x)| > \alpha\}| \leq \left|\{x||T_K(g)(x)| > \frac{\alpha}{2}\}\right| + \left|\{x||T_K(b)(x)| > \frac{\alpha}{2}\}\right|$$

We will estimate the two pieces separately. We know from Theorem 1 that

$$\|g\|^2_{L^2(\mathbb{R}^n)} \leq (1 + 2^{2n})\alpha\|f\|_{L^1(\mathbb{R}^n)}$$

and therefore

$$\begin{array}{l}\left|\{x||T_K(g)(x)| > \frac{\alpha}{2}\}\right| \leq \frac{4}{\alpha^2}\|T_K(g)\|^2_{L^2(\mathbb{R}^n)} \\ \leq cB^2\alpha^{-2}\|g\|^2_{L^2(\mathbb{R}^n)} \leq cB^2\alpha^{-1}\|f\|_{L^1(\mathbb{R}^n)}\end{array}$$

It remains to bound the term involving b. In order to do so, we will encase the cubes Q_j of the Calderon-Zygmund decompostion in concentric, larger cubes Q_j^* of diameters $2\sqrt{n}$ times larger. We consider

$$\Omega^* = \cup_j Q_j^*$$

and $F^* = \mathbb{R}^n \setminus \Omega^*$. We note that

$$|\Omega^*| \le \sum_j |Q_j^*| \le \gamma \sum_j |Q_j| = |\Omega| \le \gamma \alpha^{-1} \|f\|_{L^1(\mathbb{R}^n)}$$

holds with a constant γ depending only on n. Note also that, if we denote y_j the common center of Q_j and Q_j^*, then if $x \notin Q_j^*$, then $|x - y_j| \ge 2|y - y_j|$ holds for all $y \in Q_j$, in other words, the distance from x to the center of Q_j is larger than twice the radius of the concentric ball circumscribing Q_j. We write

$$b(x) = \sum b_j(x)$$

where $b_j(x) = b(x)$ if $x \in Q_j$, $b_j(x) = 0$ otherwise. Because the interiors of Q_j are mutually disjoint and because $b(x) = 0$ on F, for almost all x the sum reduces to one term only. Note that

$$T_K(b_j)(x) = \int_{Q_j} K(x-y) b(y) dy = \int_{Q_j} (K(x-y) - K(x-y_j)) b(y) dy$$

because $\int_{Q_j} b(y)dy = 0$. We are going now to estimate the L^1 norm of $T_K(B)$ in $F^* = \cap_j (\mathbb{R}^n \setminus Q_j^*)$.

$$\begin{array}{c}
\int_{F^*} |T_K(b)(x)| dx \le \sum_j \int_{F^*} |T_K b_j(x)|\, dx \\
\le \sum_j \int_{x \notin Q_j^*} dx \int_{y \in Q_j} |K(x-y) - K(x-y_j)||b(y)| dy \\
= \sum_j \int_{y \in Q_j} |b(y)| \left[\int_{x \notin Q_j^*} |K(x-y) - K(x-y_j)| dx \right] dy \\
\le \sum_j \int_{Q_j} |b(y)| \left[\int_{|x-y_j| \ge 2|y-y_j|} |K(x - y_j - (y - y_j)) - K(x-y_j)| dx \right] dy \\
\le B \sum_j \int_{Q_j} |b(y)| dy = B \|b\|_{L^1(\mathbb{R}^n)} \le \gamma B \|f\|_{L^1(\mathbb{R}^n)}.
\end{array}$$

Thus, by Chebyshev,

$$\left| \{x \in F^* \mid \ |T_K(b)(x)| > \frac{\alpha}{2} \} \right| \le 2\alpha^{-1} \int_{F^*} |T_K(b)(x)| dx \le 2\gamma \alpha^{-1} B \|f\|_{L^1(\mathbb{R}^n)}$$

The rest of the set where $|T_K(b)(x)| > \frac{\alpha}{2}$ is a subset of Ω^*, and therefore its measure is bounded above by the measure of Ω^*, and that is bounded by $\gamma \alpha^{-1} \|f\|_{L^1(\mathbb{R}^n)}$. This concludes the proof of the fact that T_K is weak type (1,1). The boundedness in L^2 together with the Marcinkiewicz interpolation theorem then imply the result for $1 < p < 2$. On the other hand the adjoint of T_K is computed convolving with $K(-x)$, which is a kernel that satisfies the properties in the theorem. Therefore the adjoint $(T_K)^*$ is bounded in $L^q(\mathbb{R}^n)$, $1 < q < 2$ and thus T_K is bounded in $L^p(\mathbb{R}^n)$, $2 < p < \infty$. This concludes the proof of the theorem.

Theorem 4 *Let K satisfy (1-3). Let $1 < p < \infty$ and consider the operators*

$$T_\epsilon(f)(x) = \int_{|y| \geq \epsilon} f(x-y)K(y)dy.$$

There exists a constant C_p depending on p, n and B only such that

$$\|T_\epsilon f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}$$

holds uniformly for all $\epsilon > 0$. Moreover, for each $f \in L^p(\mathbb{R}^n)$ the strong limit

$$T_K(f) = \lim_{\epsilon \to 0} T_\epsilon(f)$$

exists in the norm of $L^p(\mathbb{R}^n)$. The operator T_K is bounded in $L^p(\mathbb{R}^n)$ and obeys the same norm bound as T_ϵ.

Proof. The operators T_ϵ are convolution operators with kernels $C_\epsilon(K)$. In view of Proposition 1 and Lemma 1, these kernels satisfy the assumptions of Theorem 2, with uniform constant B. Therefore they are uniformly bounded in each $L^p(\mathbb{R}^n)$, independently of ϵ. It is then enough to check the convergence on a dense subset of $L^p(\mathbb{R}^n)$. Let $f \in C_0^\infty(\mathbb{R}^n)$. Then

$$(T_\epsilon f)(x) = \int_{|y| \geq 1} K(y) f(x-y)dy + \int_{\epsilon < |y| \leq 1} K(y)(f(x-y) - f(x))dy.$$

The first integral is a fixed function in $L^p(\mathbb{R}^n)$ because $C_1(K)$ is in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and f is in $L^1(\mathbb{R}^n)$. The limit as $\epsilon \to 0$ of the second functions is strong in $L^p(\mathbb{R}^n)$ because they are all supported in a fixed compact and the convergence is uniform. This concludes the proof of the theorem.