Construction of Angle Measure

Dustin J. Mitchell

December 23, 2003

1 Introduction

Within the axiomatic system of neutral geometry, we would like to establish a measure on angles which satisfies the intuitive properties of comparison, addition, and subtraction of angles. With a measure based in neutral geometry, the behavior of angles in Euclidean and non-Euclidean geometries can be compared and quantified.

In this construction, angles are assigned a real number between 0 and 1. Using this construction, any other unit of measurement can be substituted with a simple linear transformation of the measures. Thus an the measure of an angle $\measuredangle A$ can be expressed in radians as $\pi(\measuredangle A)^{\circ}$, and in degrees as $180(\measuredangle A)^{\circ}$.

This construction has the following structure: we begin by establishing a relationship between real numbers and their expressions in binary notation (Section 2). After introducing a few definitions and theorems to allow us to operate geometrically on a semicircle (Section 3), we construct the real number that corresponds to a given angle (Section 4). Using this construction, we then construct the angle that corresponds to a given real number in $(0, 1)$, and show that the relationship is one-to-one (Section 5). Finally, we present and prove a theorem stating that angle measure exists and is unique, and provides some useful properties (Section 6).

2 Binary Expansions

To mediate the translation of angles to real numbers and back, we define *binary expansions* and a function $R : \{b_i\} \rightarrow \mathbb{R}_{[0,2)}$ from binary expansions to $\mathbb{R}_{[0,2)}$. We also define an addition operation and a weak ordering. As these are not geometric statements, their proof is omitted.

Definition 2.1 *A finite binary expansion is a finite sequence $\{b_{0\ldots n}\}$ with $b_i \in \{0, 1\}$. Let*

$$R(\{b_{0\ldots n}\}) = \sum_{i=0}^{n} \frac{b_i}{2^i}.$$

Definition 2.2 *A binary expansion is an infinite sequence* $\{b_{0\ldots\infty}\}$ *with* $b_i \in \{0,1\}$*. Let*

$$R(\{b_i\}) = \sum_{i=0}^{\infty} \frac{b_i}{2^i}.$$

Proposition 2.1 *Finite binary expansions are a subset of binary expansions, with the following correspondance:*

$$R(\{b_{0\ldots n}\}) = R(\{b_0, \ldots, b_n, 0, 0, \ldots\}).$$

Proposition 2.2 *R is onto*

Proposition 2.3

$$R(\{b_0, \ldots, b_n, 0, 1, 1, \ldots\}) = R(\{b_0, \ldots, b_n, 1\}).$$

Proof: This proposition is a direct consequence of the summation

$$\sum_{i=0}^{\infty} \frac{1}{2^i} = 1.$$

■

Definition 2.3 *A binary expansion* $\{b_i\}$ *with* $b_0 = 0$ *is called a unit binary expansion.*

Note that unit binary expansions map to $[0,1)$. The use of larger binary expansions allows a more natural expression of the following definition.

Definition 2.4 *Given* $\{a_i\}, \{b_i\}$ *unit binary expansions, define addition of binary expansions* $\{a_i\} + \{b_i\} = \{d_i\}$ *as follows: let the "carry bit"* $c_i = 1$ *if more than one of* $a_{i+1}, b_{i+1}, c_{i+1}$ *are* 1*, otherwise let* $c_i = 0$*. Then let* $d_i = 1$ *if one or three of* a_i, b_i, c_i *are* 1*, otherwise let* $d_i = 0$*.*

Proposition 2.4 *If* $\{a_i\} + \{b_i\} = \{d_i\}$ *then* $R(\{a_i\}) + R(\{b_i\}) = R(\{d_i\})$

Definition 2.5 $\{a_i\} > \{b_i\}$ *if there is an* $n \in \mathbb{N}$ *such that for all* $1 \leq i < n$*,* $a_i = b_i$*, and both* $a_n = 1$ *and* $b_n = 0$*.*

Proposition 2.5 $\{a_i\} > \{b_i\}$ *iff* $R(\{a_i\}) > R(\{b_i\})$*.*

3 Supporting Definitions and Propositions

We will use the construction shown in Figure 1, constructed as follows: let $\overline{AC}$ be the diameter of circle γ with center B. Let $\bar{\sigma}$ be the intersection of γ with a halfplane of $\overleftrightarrow{AC}$ together with the points A and C. Let D be the intersection of $\bar{\sigma}$ with the line perpendicular to $\overleftrightarrow{AC}$ through B. Let $ADC = \overline{AD} \cup \overline{DC}$.

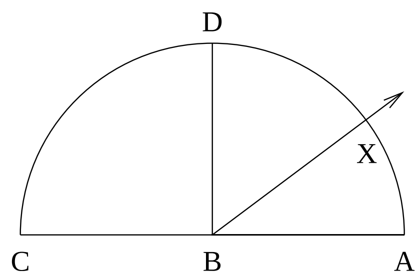

Figure 1: The standard construction

Definition 3.1 *Given three points* P, Q, R *on* $\bar{\sigma}$, $P\#Q\#R$ *if the points are distinct and* $\overrightarrow{BQ}$ *is interior to* $\measuredangle RBP$. *As a special case,* $A\#P\#B$ *for* $P \in \bar{\sigma}$.

Note the following properties of this betweenness relationship.

Proposition 3.1 *Given* $P\#Q\#R$,

- P *and* Q *are on the same side of* $\overleftrightarrow{BR}$;
- Q *and* R *are on the same side of* $\overleftrightarrow{BP}$; *and*
- P *and* R *are on opposite sides of* $\overleftrightarrow{BQ}$.

Proof: All three are direct consequences of the crossbar theorem. ■

Proposition 3.2 *Given* P, Q, R *distinct points on* $\bar{\sigma}$, *exactly one of* $P\#Q\#R$, $P\#R\#Q$ *and* $Q\#P\#R$ *holds.*

Proof: We first show that one of the relationships must hold. In so doing, we must consider the endpoints A and B as special cases.

If both $P = A$ and $R = B$, $P\#Q\#R$ by the special case in Definition 3.1 and we are done.

If only one of the points is A or B (without loss of generality, $P = A$ and $Q, R \neq B$), then Q and R are on the same side of $\overleftrightarrow{BA}$. Consider the relationship of Q and P to $\overleftrightarrow{BR}$. If they are on the same side, then Q is interior to $\measuredangle PBR$ and $P\#Q\#R$. If they are on opposite sides, then by converse to the crossbar theorem, $P\#R\#Q$.

Having handled all cases involving A and B, we limit our concern to $\bar{\sigma} - \{A, B\}$. Consider the relationship of P and R to $\overleftrightarrow{BQ}$. If the are on opposite sides, then $\overline{PR}$ intersects $\overrightarrow{BQ}$ (because P, Q, R are all on the same side of $\overleftrightarrow{AB}$; this justification will be used again without comment), so by the converse to the crossbar theorem, $P\#Q\#R$. If they are on the same side, then consider the relationship of Q and R to $\overleftrightarrow{BP}$. If these points are on the same side, then $P\#R\#Q$ immediately. If these points are on opposite sides, then $\overline{RQ}$ intersects $\overrightarrow{BP}$, so by converse to the crossbar theorem, $R\#P\#Q$.

We have shown that at least one of the three relationships must hold; we now assume that one (arbitrarily, $P\#Q\#R$) holds and show that the other two ($P\#R\#Q$ and $Q\#P\#R$) cannot. If this is the case, then Q and R are on the same side of $\overleftrightarrow{BP}$, so by Proposition 3.1 we cannot have $Q\#P\#R$. Similarly, Q and P are on the same side of $\overleftrightarrow{BR}$, so $P\#R\#Q$ is impossible. ■

Proposition 3.3 **(a)** *Given $P\#Q\#R$ and $P\#R\#S$, then $Q\#R\#S$ and $P\#Q\#S$.*

(b) *Given $P\#Q\#R$ and $Q\#R\#S$, then $P\#Q\#S$ and $P\#R\#S$.*

Proof:

(a) P and Q are on the same side of $\overleftrightarrow{BR}$ by definition, and P and S are on opposite sides of the same line by Proposition 3.1. Thus Q and S are on opposite sides of $\overleftrightarrow{BR}$, so by converse to the crossbar theorem, $Q\#R\#S$.

By definition of $Q\#R\#S$, R and S are on the same side of $\overleftrightarrow{BQ}$. By the Proposition 3.1, P and R are on opposite sides of the same line. Thus P and S are on opposite sides of $\overleftrightarrow{BQ}$, so by the converse to the crossbar theorem, $P\#Q\#S$.

(b) P and R are on opposite sides of $\overleftrightarrow{BQ}$, and R and S are on the same side of the same line, both by Property 3.1. Then P and S are on opposite sides of $\overleftrightarrow{BQ}$, so $P\#Q\#S$. $P\#R\#S$ is symmetrical.

■

We end this section by defining closed arcs on $\bar{\sigma}$. Note that there is a one-to-one relation between closed arcs (e.g., $\widehat{PR}$) and angles ($\measuredangle PBR$).

Definition 3.2 *A closed arc $\widehat{PR}$ on $\bar{\sigma}$ is $\{Q \in \bar{\sigma} | P\#Q\#R\} \cup \{P, R\}$. P and R are called the endpoints of $\widehat{PR}$.*

Proposition 3.4 *If $P\#Q\#R$, then $\widehat{PQ} \cup \widehat{QR} = \widehat{PR}$.*

Proof: Given a point S on $\widehat{PR}$ such that $P\#S\#R$, note that S must be on the same side of $\overleftrightarrow{BQ}$ as either P or R. Assume through symmetry that P and S are on the same side of this line. Consider $\overleftrightarrow{BR}$. By $P\#Q\#R$, Q and R are on the same side of this line, and by $P\#S\#R$, S and R are also on the same side of this line. Q and S, then, are on the same side of $\overleftrightarrow{BP}$. Thus we have S interior to $\measuredangle PBQ$, so $P\#S\#Q$. ■

3.1 Dedekind's Axiom on $\bar{\sigma}$

This section is taken from a proof of major exercise 4, in chapter 3 of (Greenberg, 2001).

Proposition 3.5 *There is a one-to-one and onto function $f : \bar{\sigma} \to ADC$ such that $f(P) = \overrightarrow{BP} \cap ADC$ for all $P \in \bar{\sigma}$.*

Proof: Note that all points of conern are on the same side of $\overleftrightarrow{AC}$ or on $\overleftrightarrow{AC}$; we will not further clutter the argument with discussion of the other side of $\overleftrightarrow{AC}$.

To show that f is a function, we show that for any $P \in \bar{\sigma}$, there is a unique $P' = f(P)$ in ADC. In $\triangle ADC$, $\overleftrightarrow{BP}$ intersects $\overline{AC}$, so by Pasch's Theorem it must intersect $\overline{AD}$ or $\overline{DC}$ and thus ADC, and must intersect both segments only if it passes through D. $\overleftrightarrow{BP}$ cannot intersect a single segment in two places (I-1), so if it intersects both segments then the intersection is uniquely $P' = D$. Thus P' exists, and is unique so long as it is on $\overrightarrow{BP}$; as we're only concerned with one side of $\overleftrightarrow{AC}$, this is so.

To show that f^{-1} is a function, we show that for any $P' \in ADC$, there is a unique $P = f^{-1}(P') \in \bar{\sigma}$. We begin by disposing of the border cases, where $P' \in \overleftrightarrow{AC}$. Then $P' \in \{A, C\} \subset \bar{\sigma}$, so $P = P'$. Otherwise, P' is in the same halfplane of $\overleftrightarrow{AC}$ as D, so construct P on $\overrightarrow{BP'}$ such that $\overline{BP} \cong \overline{BA}$. Thus P is on $\bar{\sigma}$ and unique by construction.

Note that $f(A) = A$, $f(D) = D$, and $f(C) = C$. These will be used without comment in what follows. ■

Definition 3.3 *Given three points P', Q', R' on ADC, $P'\#Q'\#R'$ if $P\#Q\#R$ for P, Q, R the images under f^{-1} of P', Q', R'.*

Proposition 3.6 *If P', Q', R all lie on $\overline{AD}$ or all lie on $\overline{DC}$, then $P'\#Q'\#R' \Leftrightarrow P' \star Q' \star R'$.*

Proof: Let P, Q, R in $\bar{\sigma}$ be the images under f^{-1} of P', Q', R', respectively.

To show the rightward implication, assume $P'\#Q'\#R'$. Then $P\#Q\#R$, so $\overrightarrow{BP} \star \overrightarrow{BQ} \star \overrightarrow{BR}$. Noting that $\overrightarrow{BP} = \overrightarrow{BP'}$ and so on, $\overrightarrow{BP'} \star \overrightarrow{BQ'} \star \overrightarrow{BR'}$. By the crossbar theorem, $\overrightarrow{BQ'} \cap \overline{P'R'} = \{Q'\}$, so $P' \star Q' \star R'$.

To show the converse, assume $P' \star Q' \star R'$. Then P' and Q' are on the same side of $\overleftrightarrow{BR'}$, and Q' and R' are on the same side of $\overleftrightarrow{BP'}$, so $\overrightarrow{BP'} \star \overrightarrow{BQ'} \star \overrightarrow{BR'}$ and equivalently $\overrightarrow{BP} \star \overrightarrow{BQ} \star \overrightarrow{BR}$. Then by definition $P\#Q\#R$ and thus $P'\#Q'\#R'$. ■

Theorem 3.1 (Dedekind's Axiom for $\bar{\sigma}$:) *Given two nonempty subsets Σ_1, Σ_2 of $\bar{\sigma}$ such that their disjoint union is $\bar{\sigma}$ and such that given $P\#Q\#R$, Q is in the same set as either P or R, there exists a unique point O on $\bar{\sigma}$ such that one of the subsets is equal to a closed arc and the other is the complement.*

Proof: Given the Dedekind cut Σ_i on $\bar{\sigma}$, we will construct a cut Pi_i on ADC and then a further cut Xi_i on one of the segments of ADC,

Without loss of generality, assume $A \in \Sigma_1$ and $C \in \Sigma_2$. Let $\Pi_1 = f(\Sigma_1)$ and $\Pi_2 = f(\Sigma_2)$. Because f is one-to-one and onto, these sets Π_i are nonempty, disjoint, and cover ADC.

Consider $A\#D\#C$ (the special case from Definition 3.1). Then either A and D are in Σ_1 or D and C are in Σ_2. We will handle these alternatives in turn.

- Assuming that A and D are in Σ_1, consider X' with $A \star X' \star D$, with $X = f^{-1}(X')$. $A\#X\#D$ by Proposition 3.6, so X is in the same set as A or D, that is, Σ_1. Thus $X' = f(X) \in \Pi_1$, so we have $\overline{AD} \subset \Pi_1$.

  Let $\Xi_1 = \Pi_1 \cap \overline{DC}$ and $\Xi_2 = \Pi_2 \subset \overline{DC}$. $D \in \Pi_1 \cap \overline{DC}$. so these sets are nonempty. As subsets of the Π_i, they are disjoint and cover $\overline{DC}$. Given $P' \star Q' \star R'$ with P', Q', R' in $\overline{DC}$, Proposition 3.6 gives $P\#Q\#R$ for $P = f^{-1}(P')$, $Q = f^{-1}(Q')$, $R = f^{-1}(R')$. Thus Q is in the same Σ_i as either P or R, so Q' is in the same Π_i as either P' or R', so Q' is in the same Ξ_i as P' or R'.

  We may now apply Dedekind's axiom on the segment $\overline{DC}$, producing $D \star O' \star C$ such that either $\overline{DO'} = \Xi_1$ or $\overline{O'C} = \Xi_2$. We will handle these alternatives in turn.

  - If we assume that $\overline{DO'} = \Xi_1$, then $\Pi_1 = \overline{AD} \cup \overline{DO'}$. Let $O = f^{-1}(O')$.

$$
\begin{aligned}
\Sigma_1 &= f^{-1}(\Pi_1) \\
&= f^{-1}(\overline{AD}) \cup f^{-1}(\overline{DO'}) \\
&= \{A, D, O\} \cup f^{-1}(\{X' | A \star X' \star D\}) \cup f^{-1}(\{X' | D \star X' \star O'\}) \\
&= \{A, D, O\} \cup \{X | A\#X\#D\} \cup \{X | D\#X\#O\} \\
&= \widehat{AD} \cup \widehat{DO} \\
&= \widehat{AO}.
\end{aligned}
$$

  - Assuming that $\overline{O'C} = \Xi_2$, then $\overline{O'C} = \Pi_2$. Let $D = f^{-1}(O')$.

$$
\begin{aligned}
\Sigma_2 &= f^{-1}(\overline{O'C}) \\
&= f^{-1}(\{O', C\} \cup \{X' | O' \star X' \star C\}) \\
&= \{O, C\} \cup \{X | O\#X\#C\} \\
&= \widehat{OC}.
\end{aligned}
$$

- Similarly, assuming that D and C are in Σ_2, then $\overline{DC} \subset \Pi_2$. Again we apply Dedekind's axiom on segments, producing a $A \star O' \star D$ such that either $\overline{AO'} = \Xi_1$ or $\overline{O'D} = \Xi_2$. Again, we will handle these alternatives in turn.

  - Assuming that $\overline{AO'} = \Xi_1$, $\Pi_1 = \overline{AO'}$ and $\Sigma_1 = \widehat{AO}$ for $O = f^{-1}(O')$.
  - Assuming that $\overline{O'D} = \Xi_2$, $\Pi_2 = \overline{O'D} \cup \overline{DC}$, and $\Sigma_2 = \widehat{OC}$.

Thus in all cases a Dedekind cut of $\bar{\sigma}$ produces $O \in \bar{\sigma}$ such that one of the Σ_i is a closed arc with O as an endpoint. ■

Corollary 3.1 *Given a Dedekind cut as in Theorem 3.1, if $P\#O\#R$, then P and R are in opposite sets.*

Proof: One of P or R are on the same side of $\overleftrightarrow{BO}$ as A. Without loss of generality, assume that point is R, so $A\#R\#O$.

If $O = \Sigma_1 = \widehat{AO}$, then R is in Σ_1 and by Proposition 3.3 $A\#O\#P$ so P is not in Σ_1. If $O = \Sigma_2 = \widehat{OC}$, then again by Proposition 3.3 (recalling that $A\#X\#C$ for all X in $\bar{\sigma}$), $R\#O\#C$ and thus $O\#P\#C$ so $R \notin \Sigma_2$ and $P \in \Sigma_2$. ■

3.2 Bisectors and Halves

The construction of angle measure relates binary expansions to bisections of angles. Consequently, we need to establish some properties of bisections.

Definition 3.4 *Given an angle $\measuredangle EBG$ with E and G on $\bar{\sigma}$, let $\overrightarrow{BF}$ be its bisector. Assume G and A are on the same side of $\overleftrightarrow{BF}$. Then the* left half *of $\measuredangle EBG$ is $\measuredangle EBF$ and the* right half *of $\measuredangle EBG$ is $\measuredangle FBG$. Since the two halves are congruent, we may refer to* half *of an angle when either half is acceptable.*

Proposition 3.7 *Halves of congruent angles are congruent.*

Proof: Assume the contrary, so we have congruent angles $\measuredangle BAC$ with bisector $\overrightarrow{AD}$ and $\measuredangle FEG'$ with bisector $\overrightarrow{EH}$ such that (without loss of generality) $\measuredangle FEH < \measuredangle BAD$ Construct J interior to $\measuredangle BAD$ such that $\measuredangle BAJ \cong \measuredangle FEH$. By angle subtraction, $\measuredangle CAJ \cong \measuredangle GEH$, and so by transitivity $\measuredangle CAJ \cong \measuredangle FEH$, and further $\measuredangle BAJ \cong \measuredangle CAJ$ and $\overrightarrow{AJ} \neq \overrightarrow{AD}$ bisects $\measuredangle CAB$. This violates the uniqueness of the bisector, so our assumption must be false. ■

We will measure angles using $\{\phi_i\}$, which are defined as follows: ϕ_1 is a right angle, and ϕ_{i+1} is the angle between the bisector and one leg of ϕ_i. Note that $\psi_{i+1} < \psi_i$.

Proposition 3.8 *For any angle ψ, there is an n such that $\phi_n < \psi$.*

Proof: Let $X \in \sigma$ be such that $\measuredangle ABX \cong \psi$. For each ϕ_i, let $P_i \in \sigma$ be such that $\measuredangle ABP_i \cong \phi_i$.

Let $\Sigma_1 \subset \sigma$ be all points P on σ such that $\measuredangle ABP > \measuredangle ABP_i$ for some i. We know D is in this set, so it is nonempty. Let Σ_2 be the complement. Assume that Σ_2 is nonempty (we will contradict this assumption later). Before applying theorem 3.1, we must show that if $P\#Q\#R$ then Q is in the same set as P or R.

Assume $Q \in \Sigma_1$ and $R \in \Sigma_2$. Then there is some P_i such that $\measuredangle ABP_i < \measuredangle ABQ$. Since $R \in \Sigma_2$, $\measuredangle ABR < \measuredangle ABP_i$. Then by transitivity of angle ordering, $\measuredangle ABR < \measuredangle ABQ$, or equivalently $\overrightarrow{BR}$ is interior to $\measuredangle ABQ$, so $QRA\#\#$. By Proposition 3.3, $P\#Q\#A$, so $\measuredangle ABQ < \measuredangle ABP$. By transitivity, then, $\measuredangle ABP_i < \measuredangle ABP$, so $P \in \Sigma_1$.

Now assume $P \in \Sigma_1$ and $Q \in \Sigma_2$. Then for all $i > n$ for some n, P_i is such that $\measuredangle ABP_i < \measuredangle ABP$. Because Q is in Σ_2, $\measuredangle ABQ < \measuredangle ABP_i$. By transitivity, then, $\measuredangle ABQ < \measuredangle ABP$. Trivially, $\measuredangle ABR < \measuredangle ABP_j$ for $1 \leq j \leq n$.

We would like to have $\measuredangle ABR < \measuredangle ABQ$; assume the contrary: $\measuredangle ABR > \measuredangle ABQ$. Then A is on the opposite side of $\overleftrightarrow{BQ}$ from both P and R, so P and R are on the same side of $\overleftrightarrow{BQ}$, contradicting $P\#Q\#R$. Thus $\measuredangle ABR < \measuredangle ABQ$ and by transitivity $\measuredangle ABR < \measuredangle ABP_i$ for all P_i, so $R \in \Sigma_2$.

Having verified the proper betweenness relationship for Σ_1 and Σ_2, we apply a Dedekind cut, resulting in O such that either Σ_1 or Σ_2 is a closed arc with endpoint O.

Assume the fomer; then $O \in \Sigma_1$. If $O = P_i$ for some i, then there is an X between O and P_{i+1} also in Σ_1. On the other hand, if $O \neq P_i$, then there is a point $O\#X\#P_i$ also in Σ_1. Either option is impossible, so $O \notin \Sigma_1$. Assume, then, that $O \in \Sigma_2$. If $O = P_i$ for some i, then $\measuredangle ABO > \measuredangle ABP_{i+1}$ so O is in Σ_1. Thus $O \in \Sigma_2$ and $O \neq P_i$ for any i.

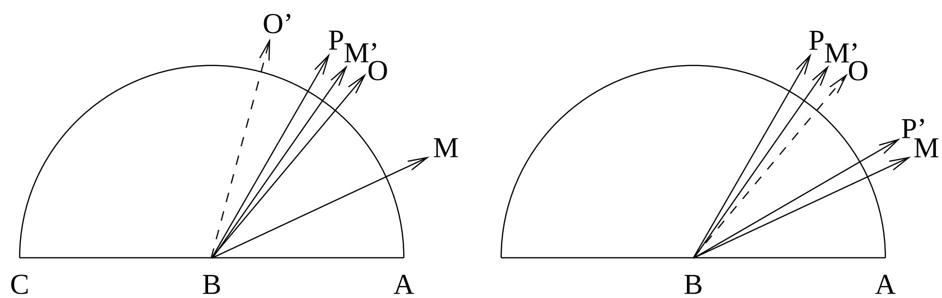

Figure 2: Construction for Proposition 3.7.

Let $\overrightarrow{BM}$ be the bisector of $\measuredangle ABO$. Pick O' on the opposite side of $\overleftrightarrow{BO}$ from M such that $\measuredangle ABM \cong \measuredangle OBO'$ (this is possible because $\measuredangle ABO$ must be acute to be less than P_0). O' is in Σ_1, so there is a P_i (call it P) such that $O'\#P\#O$. Let $\overrightarrow{BM'}$ be the bisector of $\measuredangle OBP$; see Figure 2.

Using angle addition, we will "swap" $\measuredangle MBO$ and $\measuredangle M'BO$. Construct $\overrightarrow{BP'}$ on the opposite side of $\overrightarrow{BM}$ from A such that $\measuredangle OBM' \cong \measuredangle P'BM'$. Because $\measuredangle OBM' < \measuredangle OBP < \measuredangle OBO'$, $M\#P'\#O$. By angle subtraction, $\measuredangle P'BM' \cong \measuredangle OBM$, which is congruent to $\measuredangle ABM$ because $\overrightarrow{BM}$ bisects $\measuredangle ABO$. $\overrightarrow{BM'}$ bisects $\measuredangle OBP$, so $\measuredangle MBP' \cong \measuredangle M'BP$. Thus $\measuredangle ABP' \cong \measuredangle P'BP$ and P' is a bisector of P that is interior to $\measuredangle ABO$, which violates the definition of O.

So our assumption that Σ_2 is empty is false. Thus $X \in \Sigma_1$, so there is a P_i such that $\measuredangle ABP_i < \measuredangle ABX$, and by congruence $\psi < \phi_i$. ■

4 Angles to Binary Expansions

Given an angle $\measuredangle ABX$, we will construct a binary expansion $\{b_i\}$. If $\measuredangle ABX$ is right, let $\{b_i\} = \{0, 1, 0, 0, \ldots\}$. Otherwise, let $b_0 = 0$. If $\measuredangle ABX$ is acute, let $b_1 = 0$ and $\phi_1 = \measuredangle ABD$. Otherwise, let $b_1 = 1$ and $\phi_1 = \measuredangle DBC$. In either case, X is interior to ϕ_1.

Repeat the following for each $i \in \mathbb{N}$, beginning with $i = 2$:

- if $\overrightarrow{BX}$ is the bisector of ϕ_{i-1}, let $b_i = 1$ and complete the procedure with the finite binary expansion $\{b_0, \ldots, b_i\}$.
- Otherwise, let $\overrightarrow{BM}$ be the bisector of ϕ_{i-1}. X is in one of the halfplanes of $\overleftrightarrow{BM}$, and is interior to ϕ_{i-1}, an so must be in its left or right half. Let ϕ_i be the half containing X. If it is in the left half, let $b_i = 1$; otherwise let $b_i = 0$. Note that X is interior to ϕ_i.

It is clear from this construction that X is in all ϕ_i for $i \in \mathbb{N}$.

Given the binary expansion $\{b_i\}$ corresponding to $\measuredangle ABX$, we now define $(\measuredangle ABX)^\circ = R(\{b_i\})$.

4.1 Consequences of Construction

This construction has several immediate consequences which will be useful in the next section.

Proposition 4.1 $\measuredangle ABX > \measuredangle ABY \Rightarrow (\measuredangle ABX)^\circ > (\measuredangle ABY)^\circ$.

Proof: Let ϕ_i^X be the bisected angles from the construction of $(\measuredangle ABX)^\circ$, and ϕ_i^Y be those for $(\measuredangle ABY)^\circ$. Note that $\phi_i^X \cong \phi_i^Y$ for all i, because the halves of congruent angles are congruent (Proposition 3.7). Also note that $\overrightarrow{BX} \in \phi_i^X$ and $\overrightarrow{BY} \in \phi_i^Y$ for all i, as described in the preceding construction.

Is it possible that $\phi_i^X = \phi_i^Y$ for all i? If this were so, then $\phi_i^X > \measuredangle XBY$ for all i, which contradicts Proposition 3.8. Thus there is some n such that $\phi_n^X \neq \phi_n^Y$ and the same is true for all $i > n$. By properties of $\mathbb{N}$, there is a smallest such n, such that $\phi_i^X = \phi_i^Y$ for all $i < n$ and not equal for $i \geq n$. Then by necessity ϕ_n^X and ϕ_n^Y are opposite halves of ϕ_{n-1}^X. Then by Proposition 3.3, ϕ_n^X is the left half and ϕ_n^Y the right.

The binary expansions $\{x_i\}$ and $\{y_i\}$ for $(\measuredangle ABX)^\circ$ and $(\measuredangle ABY)^\circ$, then, have $x_i = y_i$ for $i < n$ and $x_n > y_n$, so $R(\{x_i\}) > R(\{y_i\})$ and $(\measuredangle ABX)^\circ > (\measuredangle ABY)^\circ$. ■

The following proposition introduces a weak form of angle addition in which there is no carrying in the binary expansions ($c_i = 0$ for all i). This hypothesis will be strengthened in Section 6.

5 Binary Expansions to Angles

We begin by showing that an angle can be constructed for every diadic number; that is, to every fininte binary expansion $\{x_i\}$ there corresponds an angle with measure $R(\{x_i\})$.

Proposition 5.1 *For any unit finite binary expansion $\{x_0 = 0, \ldots, x_n\}$, there is an angle $\measuredangle ABO$ such that $(\measuredangle ABO)^\circ = R(\{x_0, \ldots, x_n\})$.*

Proof: We will make an inductive argument based on the length of the finite binary expansion. First, we must temporarily relax the requirement that an angle have two legs—we will consider $\measuredangle ABA$ to be an angle of measure 0, corresponding to the binary expansion $\{0, 0\}$. Likewise, $\measuredangle ABD$ corresponds to $\{0, 1\}$. It is clear that the measurement construction for each of these angles would yield the correct binary expansion, so the hypothesis holds for the base case.

For the inductive step, we are given an angle $\measuredangle ABO_{n-1}$ corresponding to $\{x_0, \ldots, x_{n-1}\}$ and x_n. If the given $x_n = 0$, let $O_n = O_{n-1}$ and note that $(\measuredangle ABO_n)^\circ = (\measuredangle ABO_{n-1})^\circ = R(\{x_0, \ldots, x_n = 0\})$, so the hypothesis holds for the inductive case.

Alternately, if $x_n = 1$, construct O_n on the opposite side of $\overleftrightarrow{BO_{n-1}}$ from A such that $\measuredangle O_{n-1}BO_n \cong \phi_n$. Because the given binary expansion for $\measuredangle ABO_{n-1}$ has no nth value nor any subsequent values, $\overrightarrow{BO_{n-1}}$ is the right leg of the ϕ_{n-1} used in the construction of its measure. By construction, then, $\overrightarrow{BO_n}$ is in the bisector of the same ϕ_{n-1}, so the process of constructing $(\measuredangle ABO_n)^\circ$ completes with the finite binary expansion $\{x_0, \ldots, x_n\}$. $(\measuredangle ABO_n)^\circ = R(\{x_0, \ldots, x_n = 0\})$, and again the hypothesis holds.

Thus the inductive step is completed, and the hypothesis holds for any n. ■

Proposition 5.2 *For $x \in (0, 1)$, there is an angle $\measuredangle ABO$ such that $(\measuredangle ABO)^\circ = x^\circ$ (denoted $\measuredangle x^\circ$).*

Proof: We may safely assume that x is not diadic; if it is, Proposition 5.1 provides the needed $\measuredangle ABO$ and we are done. Otherwise, let

$$\Sigma_1 = \{S \in \sigma | (\measuredangle ABS)^\circ > x\} \cup \{C\}$$

and Σ_2 be its complement in $\bar{\sigma}$ (so $A \in \Sigma_2$). We must show the proper betweenness properties to make a Dedekind cut. Let $P\#Q\#R$, and assume without loss of generality that R and A are on the same side of $\overleftrightarrow{BQ}$. Then $\measuredangle ABP > \measuredangle ABQ > \measuredangle ABR$ and by Proposition 4.1 $(\measuredangle ABP)^\circ > (\measuredangle ABQ)^\circ > (\measuredangle ABR)^\circ$.

If $Q \in \Sigma_1$ then $(\measuredangle ABP)^\circ > (\measuredangle ABQ)^\circ > x$ so $(\measuredangle ABP)^\circ > x$ and $P \in \Sigma_1$. Alternately, if $Q \in \Sigma_2$, then $x \geq (\measuredangle ABQ)^\circ > (\measuredangle ABR)^\circ \leq x$, so $x > (\measuredangle ABR)^\circ$ and $R \in \Sigma_2$.

We may now apply a Dedekind cut, and find $O \in \bar{\sigma}$ such that (by Corollary 3.1) for $P\#O\#R$, P and R are in opposite sets Σ_i.

We claim that $(\measuredangle ABO)^\circ = x$. Assume the contrary, so either $(\measuredangle ABO)^\circ > x$ or $x > (\measuredangle ABO)^\circ$. Let $\{x_i\}$ be the binary expansion for x, and $\{o_x\}$ the binary expansion in the construction of $(\measuredangle ABO)^\circ$.

$x > (\measuredangle ABO)^\circ$**:** We have $\{o_i\} < \{x_i\}$ by Proposition 4.1, so there is some h such that $o_h = 0$ and $x_h = 1$, and the two binary expansions are identical up to index h. Let j be the index of the first zero in $\{o_{h+1}, \ldots\}$. Note that $\{x_i\} > \{o_0, \ldots, o_{j-1}, 1\} > \{o_i\}$. Construct $\measuredangle ABM$ such that $(\measuredangle ABM)^\circ = R(\{o_0, \ldots, o_{j-1}, 1\})$. M is in the left half or on the bisector of ϕ_{j-1}, and O is in its right half (where ϕ_i are the nested halves used in the construction of $(\measuredangle ABO)^\circ$), so we have $A\#O\#M$, but all three points are in Σ_2, violating the property of O as a Dedekind cut.

We must dispense with the case where $\{o_{h+1}, \ldots\}$ are all one. Recalling that $o_h = 0$, we calculate $(\measuredangle ABO)^\circ = R(\{o_i\})$. By Proposition 2.3, $R(\{o_i\}) = R(\{o_0, \ldots, o_{h-1}, 1\} = R(\{x_i\}) = x$, contradicting the assertion that $(\measuredangle ABO)^\circ \neq x$.

$(\measuredangle ABO)^\circ > x$**:** We have $\{o_i\} > \{x_i\}$ by Proposition 4.1, so there is some h such that $o_h = 1$ and $x_h = 0$, and the two binary expansions are identical up to index h. In the construction of $(\measuredangle ABO)^\circ$, then, O is in the left half of ϕ_{h-1}. Let M be the bisector of ϕ_{h-1}, so that $(\measuredangle ABM)^\circ = R(\{o_0, \ldots, o_h = 1\}) > x$. Assuming $O \neq M$, we have $C\#O\#M$ with all three points in Σ_1, violating the property of O as a Dedekind cut.

If $O = M$, then let j be the index of the first zero in $\{x_{h+1}, \ldots\}$. Note that $\{o_i\} > \{x_0, \ldots, x_{j-1}, 1\} > \{x_i\}$. Using Proposition 5.1, construct $\measuredangle ABM$ such that $(\measuredangle ABM)^\circ = R(\{x_0, \ldots, x_{j-1}, 1\})$. Again we have $C\#O\#M$ with all three points in Σ_1, violating the property of O as a Dedekind cut.

The final case we must dispense with is the case where $\{x_{h+1}, \ldots\}$ are all one. Recalling that $x_h = 0$, we calculate $R(\{x_i\})$. By Proposition 2.3, $R(\{x_i\}) = R(\{x_0, \ldots, x_{h-1}, 1\} = R(\{o_i\}) = (\measuredangle ABO)^\circ$, contradicting the assertion that $(\measuredangle ABO)^\circ \neq x$.

Thus we have shown that $(\measuredangle ABO)^\circ \neq x$ always leads to a contradiction, so we conclude that $(\measuredangle ABO)^\circ = x$. ■

5.1 Addition of Measures

We wish to show that two angles added in the geometric sense (via the angle addition axiom) to a larger angle have measures which sum to the measure of the larger angle. Before approaching this proposition, however, we establish the following propositions. the first allows us to split an angle into a diadic angle (the measure of which is a finite binary expansion) and a "remainder" angle. The second is a weak form of addition of measures, limited to diadic angles.

Proposition 5.3 *Given an angle $\measuredangle ABX$ with binary expansion $\{x_i\}$, for any $n \in \mathbb{N}$ we may construct $\measuredangle ABX_n$ such that $(\measuredangle ABX_n)^\circ = R(\{x_0, \ldots, x_n\})$ and either $x_i = 0$ for all $i > n$ and $X_n = X$ or*

$$(\measuredangle X_n BX)^\circ = R(\{\overbrace{0, \ldots, 0}^{n\ 0'\text{s}}, x_{n+1}, \ldots\}).$$

We will call the former angle the nth diadic subangle of $\measuredangle ABX$ and $\measuredangle X_nBX$ its remainder.

Proof: Construct $\measuredangle ABX_n = \measuredangle(R(\{x_0, \ldots, x_n\}))^\circ$. If $x_i = 0$ for all $i > n$ then $X = X_n$ by (iv), above, and we are finished. Otherwise, we examine the construction of $(\measuredangle X_nBX)^\circ$, yielding a binary expansion $\{y_i\}$ and using angles ψ_i.

For $i \leq n$, X and X_n are both in ϕ_i, as both have x_i in the ith place in their respective binary expansions. Thus $\measuredangle X_nBX < \phi_i$ for all such i, so $y_i = 0$. For $i = n$, note that ϕ_n has $\overrightarrow{BX_n}$ as its right leg (the leg bordering its right half). Because the first n y_i are zero, ψ_n also has $\overrightarrow{BX_n}$ as its right leg. By Proposition 3.7, then, $\phi_i = \psi_i$. As these are the only variables carried to the next stage in the constructions of $(\measuredangle ABX)^\circ$ and $(\measuredangle X_nBX)^\circ$, respectively, the constructions will produce identical binary expansions for $i > n$, so $y_i = x_i$ for all such i and we are done. ■

Proposition 5.4 *Given $A\#P\#Q$ with $\measuredangle ABP$ and $\measuredangle PBQ$ both diadic of degree $\leq n$ (having finite binary expansions of length $\leq n$ as their measure), $(\measuredangle ABQ)^\circ = (\measuredangle ABP)^\circ + (\measuredangle PBQ)^\circ$ and is thus diadic of the same degree.*

Proof: We apply the definition of binary addition, with $\{p_0, \ldots, p_n\}$ and $\{q_0, \ldots, q_n\}$ the finite binary expansions for $(\measuredangle ABP)^\circ$ and $(\measuredangle PBQ)^\circ$, respectively. We consider $\measuredangle ABP$ to be the geometric sum of ϕ_i for all i such that $p_i = 1$, and likewise $\measuredangle PBQ$ is the geometric sum of ϕ_i for all i such that $q_i = 1$. These sums are finite, and thus manageable with the machinery of the angle addition axiom; we speak informally only for the sake of clarity in the argument. We further employ the angle addition axiom to rearrange these angles in order from smallest to largest; their sum remains $\measuredangle ABQ$. We now turn to the binary expansion for $(\measuredangle ABQ)^\circ$, $\{r_i\}$, and simultaneously to the sum $\{d_i\} = \{p_i\} + \{q_i\}$ (we will also use the carry bits $\{c_i\}$ in this sum).

For $i > n$, we have $p_i = q_i = c_i = 0$, so $d_i = 0$; thus $c_n = 0$. For each $i \leq n$, beginning with $i = n$, consider the following: within the set of angles under consideration, there may be from zero to three ϕ_i's; likewise, there may be from zero to three 1's in $\{p_i, q_i, c_i\}$, and in fact there are the same number of each. If there are two or more ϕ_i's, combine two of them into a ϕ_{i-1} (as the ϕ_i are halves of this angle) and save it for consideration in the next cycle; likewise, in exactly this case $c_{i+1} = 1$. If and only if one ϕ_i remains (e.g., we began with one or three of them) then we will also have $d_i = 1$.

Thus $\measuredangle ABQ$ is the geometric sum of ϕ_i where $d_i = 1$, so

$$(\measuredangle ABQ)^\circ = R(\{d_i\}) = R(\{p_i\} + \{q_i\}) = (\measuredangle ABP)^\circ + (\measuredangle PBQ)^\circ$$

as hypothesized. ■

Proposition 5.5 *Given $A\#X\#Y$ and binary expansions $\{x_i\}$ and $\{y_i\}$ such that $(\measuredangle ABX)^\circ = R(\{x_i\})$ and $(\measuredangle XBY)^\circ = R(\{y_i\})$, $(\measuredangle ABY)^\circ = R(\{x_i\} + \{y_i\})$.*

Proof: We begin by defining P_i and Q_i for all i such that $A\#P_i\#X$ and $X\#Q_i\#Y$ and such that $\measuredangle ABP_i$ and $\measuredangle XBQ_i$ are the ith diadic angles of $\measuredangle ABX$ and $\measuredangle XBY$, respectively.

For each i, angle addition allows us to "swap" angles $\measuredangle P_iBX$ and $\measuredangle XBQ_i$, constructing $\overrightarrow{BX_i'}$ as shown in Figure 3. With this swap completed, $(\measuredangle ABX_i')^\circ = (\measuredangle ABP_i)^\circ + (\measuredangle X_i'BP_i)^\circ = (\measuredangle ABP_i)^\circ + (\measuredangle XBQ_i)^\circ$ (the sum of the ith diadic angles) by Proposition 5.4. X_i' is internal to $\measuredangle P_iBQ_i$ and thus internal to $\measuredangle ABY$, so $\measuredangle ABX_i' < \measuredangle ABY$. Furthermore, each of $\measuredangle Q_iBY$ and $\measuredangle Q_iBX$ (the remainders) is less than ϕ_i, so their geometric sum $(\measuredangle X_i'BY)$ must be less than ϕ_{i-1}. Geometrically, then, $\measuredangle ABX_i' + \measuredangle X_i'BY = \measuredangle ABY$, so $\measuredangle ABX_i' + \phi_{i+1} > \measuredangle ABY$.

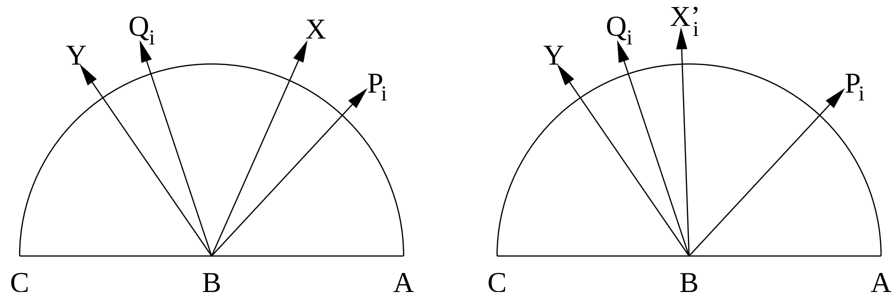

Figure 3: "Swapping" angles $\measuredangle P_iBX$ and $\measuredangle XBQ_i$.

So, for all i,

$$(\measuredangle ABX_i')^\circ < (\measuredangle ABY)^\circ < (\measuredangle ABX_i')^\circ + (\measuredangle \phi_{i+1})^\circ.$$

$(\measuredangle \phi_{i+1})^\circ = \frac{1}{2^{i+1}}$, which has limit zero as $i \longrightarrow \infty$, so we have $(\measuredangle ABY)^\circ$ in a sequence of nested intervals, the length of which is going to zero. By properties of $\mathbb{R}$, then, there is only one such value, and it is equal to the limit of the sequence of endpoints of the intervals. In this case, we will use the limit of $(\measuredangle ABX_i')^\circ = (\measuredangle ABP_i)^\circ + (\measuredangle XBQ_i)^\circ$, so (for $\{p_i\}$ and $\{q_i\}$ the binary expansions for $(\measuredangle ABX)^\circ$ and $(\measuredangle XBY)^\circ$ respectively):

$$\begin{aligned}
\lim_{i\longrightarrow\infty} (\measuredangle ABX_i')^\circ &= \lim_{i\longrightarrow\infty} (\measuredangle ABP_i)^\circ + (\measuredangle XBQ_i)^\circ \\
&= \lim_{i\longrightarrow\infty} \sum_{j=0}^{i} \frac{p_j}{2^j} + \sum_{j=0}^{i} \frac{q_j}{2^j} \\
&= \sum_{j=0}^{\infty} \frac{p_j}{2^j} + \sum_{j=0}^{\infty} \frac{q_j}{2^j} \\
&= (\measuredangle ABX)^\circ + (\measuredangle XBY)^\circ
\end{aligned}$$

as hypothesized. ■

6 Angle Measure

We many now prove the following theorem:

Theorem 6.1 *There is a unique way of assigning a degree measure to each angle such that the following properties hold:*

(i) $(\measuredangle A)^\circ$ *is a real number in* $(0,1)$*;*

(ii) *For* $x \in (0,1)$*, there is an angle* $\measuredangle A$ *such that* $(\measuredangle A)^\circ = x$*;*

(iii) $(\measuredangle A)^\circ = \frac{1}{2}$ *iff* $\measuredangle A$ *is a right angle;*

(iv) $(\measuredangle A)^\circ = (\measuredangle B)^\circ \Leftrightarrow \measuredangle A \cong \measuredangle B$*;*

(v) *If* $\overrightarrow{AC}$ *is interior to* $\measuredangle DAB$*, then* $(\measuredangle DAB)^\circ = (\measuredangle DAC)^\circ + (\measuredangle CAB)^\circ$*;*

(vi) *If* $\measuredangle B$ *is supplementary to* $\measuredangle A$*, then* $(\measuredangle A)^\circ + (\measuredangle B)^\circ = 1$*; and*

(vii) $(\measuredangle A)^\circ > (\measuredangle B)^\circ$ *iff* $\measuredangle A > \measuredangle B$*.*

Proof:

(i) See Section 4; a unit binary expansion is produced, which corresponds to a real number in $(0,1)$.

(ii) See Section 5.

(iii) See the first sentence of Section 4.

(iv) Begin with the converse: given $\measuredangle A \cong \measuredangle B$ and the construction for $(\measuredangle X)^\circ$, we may produce a congruent construction for $(\measuredangle Y)^\circ$, producing the same measure. As for the forward direction, if we have two angles $\measuredangle A \ncong \measuredangle B$ with equal measures, then we may construct congruent angles $\measuredangle ABX$ and $\measuredangle ABY$ on the standard construction. By the converse just demonstrated, $(\measuredangle ABX)^\circ = (\measuredangle A)^\circ = (\measuredangle B)^\circ = (\measuredangle ABY)^\circ$, but $X \neq Y$, violating the uniqueness of the construction in Section 5.

(v) See Proposition 5.5.

(vi) Construct $X \in \bar{\sigma}$ in the standard construction such that $\measuredangle ABX \cong \measuredangle A$ and $\measuredangle XBC \cong \measuredangle B$. By symmetry in the construction of binary expansions for each angle, every left half for $\measuredangle ABX$ will be a right half of $\measuredangle XBC$ and vice versa. Thus for binary expansions $\{x_i\}$ and $\{y_i\}$ for $\measuredangle A$ and $\measuredangle B$, exactly one of x_i and y_i is 1 for each $i > 0$. Applying Definition 2.4, all $c_i = 1$, including c_0. All d_i except d_0 are thus 0, and $d_0 = 1$. Thus $R(\{x_i\}) + R(\{y_i\}) = R(\{x_i\} + \{y_i\}) = R(\{d_i\}) = 1$.

(vii) Proposition 4.1 handles the reverse implication of this statement. Thus it only remains to show that if $(\measuredangle A)^\circ > (\measuredangle B)^\circ$, then $\measuredangle A > \measuredangle B$. Assume the contrary, so that $\measuredangle A < \measuredangle B$. Then $(\measuredangle A)^\circ < (\measuredangle B)^\circ$ by Proposition 4.1, which is impossible.

■