Signal Processing

Pedro F. Felzenszwalb

A discrete signal is just a sequence of numbers, usually corresponding to the sampling of a continuous signal at equally spaced points. We can represent a discrete signal as a (possibly infinite) vector,

$$f = [\ldots, f(-1), f(0), f(1), \ldots].$$

A finite signal can be thought of as an infinite signal with value $f(i) = 0$ for all i outside the range of f.

In signal processing we study systems that transforms signals. An important class of transformations is defined by *convolutions*. Let f and g be two signals. The convolution of f with g is denoted by $f \otimes g$ and its defined to be the signal h given by,

$$h(i) = \sum_j f(j)g(i-j).$$

Intuitivelly $h(i)$ is the dot product of f with an inverted copy of g shifted by i. We usually think of g as a mask which defines a transformation taking f to $f \otimes g$. Convolution is a commutative and associative operation,

$$f \otimes g = g \otimes f,$$

$$f \otimes (g \otimes h) = (f \otimes g) \otimes h.$$

Convolution is also linear and shift-invariant as defined below. The brute force computation of the convolution of two n dimensional vectors takes $O(n^2)$ time. The fast Fourier transform (FFT) allow us to compute such convolution in $O(n \log n)$ time. In practice this has an incredible impact in many applications including audio processing, medical image analysis, image processing, pattern recognition, error correcting codes, etc.

We say that a system S is *linear* if it obeys the principle of superposition,

$$S(f+g) = S(f) + S(g), \text{ and } S(af) = aS(f),$$

where a is a scalar. We say that S is *shift-invariant* if shifting the input changes the output by a similar shift,

$$S(f(i-t)) = S(f)(i-t).$$

The most widely used (and understood) systems are *linear shift-invariant (LSI)*. The output of these systems can be seen as convolutions.

One particularly important signal is the unit impuse δ defined by

$$\delta(0) = 1, \text{ and } \delta(i) = 0 \text{ when } i \neq 0.$$

For a linear shift-invariant system S, the impulse response $r = S(\delta)$ completely characterizes the system. We can rewrite any signal f as a weighted sum of shifted impulses. Since the system is linear, the response of S to the weighted sum is just the weighted sum of the responses to each shifted impulse. Since the system is shift-invariant the response to each shifted impulse is just a shifted version of r. Its not hard to see that,

$$S(f) = f \otimes r.$$

There is a nice connection between convolutions and polynomial multiplication. Let

$$p(x) = \sum_{i=0}^{n-1} a(i)x^i, \text{ and } q(x) = \sum_{i=0}^{n-1} b(i)x^i.$$

be two polynomials of degree $(n-1)$. The product of $p(x)$ and $q(x)$ is a polynomial of degree $(2n-2)$,

$$p(x)q(x) = \sum_{i=0}^{2n-2} \left(\sum_j a(j)b(i-j) \right) x^i.$$

Note how the coefficients are exactly the components of the convolution of the coefficient vectors a and b of the original polynomials. So a fast convolution algorithm allow us to do fast polynomial multiplication.