Problems in Randomness

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1 Preliminaries

Let $\varphi_e$ denote the $e$-th (partial) computable function. By the Universal Turing Machine Theorem, there exists a single (partial) computable function $\psi(e, n) = \varphi_e(n)$.

Note that for the purposes of this assignment, we will often treat the set of natural numbers $\omega = \{0, 1, \ldots\}$ and the set of strings $\{0, 1\}^* = \{\varepsilon, 0, 1, 00, 01, \ldots\}$ as interchangeable.

2 Kolmogorov Complexity

For an arbitrary function $\xi : \{0, 1\}^* \to \{0, 1\}^*$, we can define $K_{\xi}(x)$ to be the length of the shortest $y$ such that $\xi(y) = x$.

**Problem 1** Give the simplest possible condition on $\xi$ such that $K_{\xi}(x)$ is defined for all $x \in \{0, 1\}^*$.

**Theorem 2.1 (Kolmogorov)** There exists a partial recursive function $\upsilon$ such that for all partial recursive $\xi$, there exists a constant $c_\xi$ such that

$$K_\upsilon(x) \leq K_{\xi}(x) + c_\xi,$$

for all $x \in \{0, 1\}^*$.

**Problem 2** Prove Theorem 2.1. Be explicit about coding details.

For the subsequent discussion, we assume that we’ve picked a $\upsilon$ as in Theorem 2.1, and write $K(x)$ for $K_\upsilon(x)$.

**Problem 3** Show that $K$ is total, i.e., that $K(x)$ is defined for all $x \in \{0, 1\}^*$.

**Problem 4** Show that there exists a constant $c$ such that $K(x) \leq |x| + c$ for all $x \in \{0, 1\}^*$.

**Problem 5** Show that for every $k \in \omega$, there must exist at least one string $x$ of length $k$ such that $K(x) \geq k$. Show that most strings $x$ of length $k$ have $K(x) \geq k - 1$.

**Problem 6** Show that there are infinitely many strings $x \in \{0, 1\}^*$ such that $K(x)^2 < |x|$.

**Problem 7** Problem 5 can be generalized beyond simple squaring. What is the best such theorem that you can prove?