Are Cook and Karp Ever the Same?

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Abstract

We consider the question whether there exists a set A such that every set polynomial-time Turing equivalent to A is also many-one equivalent to A. We show that if $\mathbf{E} = \mathbf{NE}$ then no sparse set has this property. We give the first relativized world where there exists a set with this property, and in this world the set A is sparse.

1 Introduction

The seminal papers of Cook [Coo71] and Karp [Kar72] define $\mathbf{NP}$-completeness differently based on the kind of reductions used. Cook allows Turing reductions, where one can decide an instance of one problem by asking an arbitrary collection of instances of a second problem. Karp allows only the more restrictive many-one reductions, which require mapping an instance of one problem into a single instance of the second. Computational complexity theory has often looked at the relationship between these two notions of reducibility.

In this paper, we will focus on a fundamental question along these lines: Does there exist a set A such that all sets Turing equivalent to A are also many-one equivalent to A? We say that such a set *collapses* or has the *collapsing* property. While there has been some work on this problem, most notably by Selman [Sel79] and by Ambos-Spies, Bentzien, Fejer, Merkle and Stephan [ABF$^+$00], three decades after the work of Cook and Karp this question remains open.

Selman [Sel79] shows that no tally set collapses. Under the assumption $\mathbf{E} = \mathbf{NE}$ we show that no sparse set collapses either.

Some assumption appears to be necessary: We give a relativized world where there *is* a sparse set that collapses. This is in fact the first relativized world in which any set has been shown to collapse.

In Section 2 we give some background and formal definitions of our problem. Section 4 shows that no sparse set collapses unless $\mathbf{E} \neq \mathbf{NE}$. Section 5 describes the relativized world giving a sparse collapsing set.

2 Preliminaries

We say a set $A \leq_m^p B$ ("A many-one reduces to B") if there exists a polynomial-time computable function f such that $x \in A \Leftrightarrow f(x) \in B$ and a set $A \equiv_m^p B$ ("A is many-one equivalent to B") if $A \leq_m^p B$ and $B \leq_m^p A$. Similarly a set $A \leq_T^p B$ ("A Turing reduces to B") if there exists a polynomial-time oracle Turing machine M such that $A = L(M^B)$, and a set $A \equiv_T^p B$ ("A is Turing equivalent to B") if $A \leq_T^p B$ and $B \leq_T^p A$.

Hypothesis 2.1 (Collapse) *There exists an A not in $\mathbf{P}$ such that for all B, if $A \equiv_T^p B$ then $A \equiv_m^p B$.*

We call a set A fulfilling Hypothesis 2.1 a *collapsing* set. The condition that A is not in $\mathbf{P}$ is not strictly necessary—for any $A \neq \emptyset$ in $\mathbf{P}$, $\emptyset$ is Turing equivalent to A, but $\emptyset$ is not many-one equivalent to A. Nevertheless we will carry along this condition for clarity.

Collapsing is invariant under Turing equivalence: If A is collapsing and B is Turing equivalent to A then B is also collapsing.

The join of two sets $A \oplus B$ is the set $\{0x \mid x \in A\} \cup \{1x \mid x \in B\}$. The set $A \oplus B$ is many-one reducible to any other set C such that A many-one reduces to C and B many-one reduces to C. A similar statement holds for Turing reductions.

A set A is *sparse* if for some polynomial p, there are at most $p(n)$ elements of A of length n, for all n. A set A is ***P**-printable* if there is a function h computable in polynomial-time such that $h(1^n)$ outputs a list of all strings in A having length n. Every $\mathbf{P}$-printable set is sparse.

The complexity class $\mathbf{E} = \mathbf{DTIME}(2^{O(n)})$, and $\mathbf{NE} = \mathbf{NTIME}(2^{O(n)})$.

We use Σ to represent our alphabet $\{0, 1\}$.

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3 Previous Work

Selman [Sel79] looked at whether collapsing sets could be tally sets, i.e., subsets of 1^*.

Theorem 3.1 (Selman) *No tally set is collapsing.*

Since we make use of Theorem 3.1 in this paper, we give a proof here for completeness.

Proof of Theorem 3.1: Suppose $A \subseteq 1^*$ is a collapsing set. Consider the following set B (known as the left-cut of A):

$$B = \{x \in \Sigma^* \mid x \leq A(\epsilon)A(1)A(11)\cdots A(1^{|x|-1})\}$$

where $\leq$ denotes lexicographic ordering.

One can easily Turing reduce B to A. A also Turing reduces to B by using binary search to find $A(\epsilon)A(1)\cdots$.

The sets B and $\overline{B}$ are both Turing equivalent to A so they are Turing equivalent to each other. By assumption this means there is a many-one reduction f reducing B to $\overline{B}$.

Note that if $x \in B$ and $y \leq x$ then $y \in B$. Since exactly one of x and $f(x)$ could be in B, it must be the lexicographically least one. This lets us test membership in B in polynomial-time. Since A is equivalent to B we also have A in $\mathbf{P}$. □

Ambos-Spies, Bentzien, Fejer, Merkle and Stephan [ABF+00] proved many other results about collapsing sets including the following:

Theorem 3.2 (ABFMS) *If A is a collapsing set then*

1. *A is computable, and*
2. *A is not hard for $\mathbf{EXP}$.*

4 Separations

In this section we show that under a hard-to-disprove assumption, no sparse set can be collapsing. Section 5 indicates that some assumption appears to be needed.

Theorem 4.1 *If $\mathbf{E} = \mathbf{NE}$ then no sparse set is collapsing.*

To prove Theorem 4.1 we first need the following lemma:

Lemma 4.2 *No subset of a $\mathbf{P}$-printable set is collapsing.*

Proof: Let B be a subset of a $\mathbf{P}$-printable set C. Let $h(1^n) = \{y_{n,1}, \ldots, y_{n,k}\}$ list the elements of C of length n. Define the set D as

$$D = \{1^{\langle n,i\rangle} \mid y_{n,i} \in B\}.$$

The set D is Turing equivalent to B so if B is collapsing then D is collapsing. But D is a tally set so D is not collapsing by Theorem 3.1. □

We also use the following lemma from Hartmanis, Immerman and Sewelson [HIS85]:

Lemma 4.3 (HIS) *If $\mathbf{E} = \mathbf{NE}$ then all sparse sets in $\mathbf{NP}$ are $\mathbf{P}$-printable.*

Proof: Let A be a sparse set in $\mathbf{NP}$. Consider the sets B and C defined as

$$\begin{aligned} B &= \{\langle n, w\rangle \mid \text{There exist } x_1 < x_2 < \cdots < x_w \\ &\quad \text{of length } n \text{ in } A\} \\ C &= \{\langle n, w, i, j\rangle \mid \text{There exist } x_1 < x_2 < \cdots < x_w \\ &\quad \text{of length } n \text{ in } A \text{ with the } j\text{th bit of } x_i \text{ is } 1\} \end{aligned}$$

where the numbers n, w, i and j are written in binary. The sets B and E are in $\mathbf{NE} = \mathbf{E}$. In particular, in polynomial in n time we can find the maximum w such that $\langle n, w\rangle$ is in B. This tells us how many strings of length n are in A. Note that there is only one possible choice for $x_1, \ldots, x_w$. We now determine whether $\langle n, w, i, j\rangle$ in C for all of the appropriate i and j to reconstruct the strings $x_1, \ldots, x_w$ in A. □

Proof of Theorem 4.1: Suppose $\mathbf{E} = \mathbf{NE}$ and let A be a sparse collapsing set. A is Turing equivalent to $\overline{A}$ so by assumption there is a function f reducing A to $\overline{A}$. Consider the set

$$B = \{\langle 1^n, x\rangle \mid (\exists y)[|y| = n \text{ and } f(y) = x\}.$$

The set B is in $\mathbf{NP}$.

We also claim that B is sparse. The number of strings in B of length n is bounded by $\sum_{m \leq n} |f(\Sigma^m)|$, so it suffices to show that $|f(\Sigma^n)|$ is polynomially bounded. We have $|f(\Sigma^n)| \leq |f(A \cap \Sigma^n)| + |f(\overline{A} \cap \Sigma^n)|$. Each of those terms is polynomially bounded because A is sparse and $f(\overline{A}) \subseteq A$.

By Lemma 4.3, B is $\mathbf{P}$-printable. Consider $C \subseteq B$ defined as

$$C = \{\langle 1^n, x\rangle \mid \langle 1^n, x\rangle \in B \text{ and } x \in A\}.$$

The set C is Turing equivalent to A: C is reducible to A because B is in $\mathbf{P}$; for the other direction, we have $y \in A \iff \langle 1^{|y|}, f(y)\rangle \notin C$.

Since A is collapsing C is also collapsing. But C is a subset of a $\mathbf{P}$-printable set so C does not collapse by Lemma 4.2. □

5 Collapse

In this section we present the first relativized world where there exists a collapsing set, i.e., where Hypothesis 2.1 holds.

Theorem 5.1 *There exists an oracle B and a sparse set $A \notin \mathbf{P}^B$ such that for every set C*

$$C \equiv_T^{p^B} A \iff C \equiv_m^{p^B} A.$$

Proof: Let $M_1, M_2, \ldots$ be an enumeration of polynomial-time oracle Turing machines. Note that $C \equiv_T^{p^B} A$ is the same as saying there exist machines M_i and M_j such that $C = L(M_i^{B \oplus A})$ and $A = L(M_j^{B \oplus C})$. Let $R_1, R_2, \ldots$ and $S_1, S_2, \ldots$ be enumerations of polynomial-time machines such that for every i and j there is a k such that $R_k = M_i$ and $S_k = M_j$. We assume without loss of generality that M_i, R_i and S_i run in time at most n^i for n the length of the input.

We need to fulfill the following requirements:

1. E_i: There is an x such that $x \in A \Leftrightarrow M_i^B(x)$ rejects.
2. F_i: If $C = L(R_i^{B \oplus A})$ and $A = L(S_i^{B \oplus C})$ then $A \leq_m^{p^B} C$ and $C \leq_m^{p^B} A$.

The E_i requirements guarantee that A is not in $\mathbf{P}^B$ while the F_i requirements guarantee that A is collapsing relative to B.

When we fulfill our F_i requirements we will allow our many-one reductions to occasionally just accept and reject. Since our E_i requirements will guarantee that A is not in $\mathbf{P}^B$ we can just map to fixed strings known to be in or out of A and C.

We will define our oracle B as a function $B : \{0,1\}^n \to \{0,1\}^{n^k}$ for some fixed k. One can use standard coding ideas to create a subset of $\{0,1\}^*$ equivalent to B.

We build the oracle in stages. In each stage s we will pick $n_s = 2^{n_{s-1}}$.

We will set $B(1^{n_{s-1}^{\log n_{s-1}}})$ to be a list of the elements of $A \cap \Sigma^{n_{s-1}}$. This guarantees that any reduction that looks at previous strings in A can just find them easily in B.

Fix s and let $n = n_s$. In stage s we will fulfill requirement E_s for some string of length n. We will also guarantee that $F_1, \ldots, F_s$ hold for strings of length between $n_{s-1}^{\log n_{s-1}}$ and $n_s^{\log n_s}$. At the end of the construction, each requirement F_i will have been fulfilled for all but a finite number of strings.

Choose a Kolmogorov random string of length n^2 and break it into n pieces of length n each and call these strings $y_1, \ldots y_n$. Let $U = \{y_1, y_1 + 1, \ldots, y_n, y_n + 1\}$ where $y_i + 1$ is the lexicographically next string after y_i. We will guarantee that $A \cap \Sigma^n$ is a subset of U.

We set $B(u) = 1$ for each u in U.

We will put conditions on A of the form $u \in A \Leftrightarrow v \notin A$. The construction will not build a contradictory set of conditions and will allow some freedom so at the end we can choose A to fulfill E_s.

We say u and v are *connected* if the value of $A(u)$ is forced by setting $v \in A$, given the other conditions we have already currently set.

The first set of conditions we use will require that $y_i \in A \Leftrightarrow y_i + 1 \notin A$ for each i.

Eventually we will connect all the members of U. For all pairs u and v of elements in U we will encode at $B(\langle u, v \rangle)$ whether $u \in A \Leftrightarrow v \in A$ or $u \in A \Leftrightarrow v \notin A$.

We will now work in substages $j = 1, \ldots, s$. In each substage we will handle requirement F_j.

In substage j and later we guarantee not to add any strings to B of length less than $n^{j^{10}}$. Thus $\mathrm{DTIME}(n^j)$ machines or even a composition of five of them acting on strings of length n will not be affected by strings we add to B.

We first describe how to construct the many-one reduction from C to A. Fix an input w and simulate $S_j^{B \oplus A}(w)$, the potential Turing reduction from C to A. If S_j makes a query to B we just query it. Now consider all of the queries made to A. Since we have encoded the relationship between all pairs u and v in U, we can reduce $S_j^{B \oplus A}(w)$ to a single query y_i in A. If S_j accepts or rejects regardless of whether y_i is in A then just accept or reject respectively. If S_j accepts iff y_i is in A then we have our many-one reduction. Otherwise reduce to $y_i + 1$.

We now describe how to construct the many-one reduction from A to C. Fix a u of length n. Check with B to see if u is in U. If not then we just reject.

Simulate $R_j^{B \oplus C}(u)$, the potential reduction from A to C. Consider the strings it queries in C in order. Let w be the next string queried and consider the reduction $S_j^{B \oplus A}(w)$. Note that $S_j^{B \oplus A}$ cannot query any string in U not connected to u or we would violate the randomness of the string used to generate the y_i's.

By the above argument we have only three possibilities. If S_j accepts or rejects regardless of A then we answer R_j accordingly and go to R_j's next query. If $S_j^{B \oplus A}(w)$ accepts iff $u \in A$ then we just output w as our many-one reduction and we then go on to the next u in U.

The problem occurs if S_j accepts iff $u \notin A$. Set $w = w_u$ and we'll handle this case later. Go on and handle the next u in U.

Suppose for some u, $R_j^{B \oplus A}(u)$ runs out of queries and just accepts or rejects. We then set u in A to diagonalize against R_j and go on to the next substage.

Now consider the w_u's. If we have a w_u and a w_v (with u connected to v) such that $u \in A \iff v \notin A$, then we can just use the many-one reduction of v to w_u and u to w_v. Otherwise pick a w_u and a w_v not connected, and add the condition $u \in A \iff v \notin A$. We encode this connection in the oracle by setting $B(\langle 1^{n^{j^{10}}}, u \rangle) = v$ and $B(\langle 1^{n^{j^{10}}}, v \rangle) = u$. We then have the many-one reduction

map all of the r with $r \in A \iff u \in A$ to w_v and all of the s with $s \in A \iff v \in A$ to w_u. We then remove from consideration all of these w_r's and w_s's and repeat.

If we have one w_u remaining without a w_v to connect to it then just set $u \in A$ and encode all of the r connected to u in B as $B(\langle 1^{n^{j^{1_0}}}, u\rangle) =$ TRUE or FALSE as appropriate and have the many-one reduction on u just accept.

At the beginning of stage s we have n connected components of elements of U. In each stage we shrink the number of components in at most half and possibly determine the elements of one component. Since $s \ll \log n$, we still have components remaining after substage s. Pick some u in one of these components and set u in A to diagonalize against the $M_s(u)$ to fulfill E_s. □

6 Further Research

Of course, the question whether there exists a collapsing set still remains open. In particular can one modify the oracle construction in Section 5 to simply create a (nonsparse) collapsing set without needing a relativized world?

Another related question is whether all Turing complete sets for $\mathbf{NP}$ are many-one complete. This question is incomparable to the existence of a collapsing set since in this case we restrict ourselves to sets in the class $\mathbf{NP}$, which is not necessarily closed under Turing reductions. Nevertheless, perhaps our techniques could also shed light on that problem.

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