1 Introduction

1.1 Motivation
As previously discussed, quantum systems are not ideal. There are many variables that have an impact on the outcome of a computation. The fidelity rates and coherence times are some factors posing challenges. Quantum gate operations and control signals are not perfect. And all of these errors build up to non-negligible amounts. That’s why we need a way to correct all the accumulating errors. This is the motivation behind quantum error correction. Simply put, the purpose of quantum error correction can be summarized as protecting quantum circuits from noise.

1.2 Quantum Error vs. Classical Error
Classically, we are using bits, so the information is stored in 0’s and 1’s. Whenever there’s an error, the bit is the opposite of what it’s supposed to be (i.e. a bit flip). Because classical errors are just accidental bit flips, they are digitized. However, quantum errors are continuous. This continuous error can be mathematically modeled as follows:

$$\left| 0 \right\rangle \xrightarrow{X \text{ gate}} \sqrt{\epsilon} \left| 0 \right\rangle + \sqrt{1 - \epsilon} \left| 1 \right\rangle,$$  \hspace{1cm} (1)

Even though physicists do their best to reduce this effect, it sometimes isn’t enough. There are a lot of questions that rise from this situation. Are we able to detect and measure how big/small this error $\epsilon$ is, or even if we can, is it better to correct it right now or later? One of the hard to questions to answer is at what point do we decide to attempt to correct $\epsilon$?

2 Key Ideas in Quantum Error Correction
There are two main ideas that make quantum error correction possible. One idea is to use redundant encoding of information, just like in QR codes. This way, affects of noise in certain parts of the system can be tolerated and will not end up corrupting the state of the system. Another main idea is to digitize quantum error, since we know how to deal with digitized errors, as they resemble the classical case.
• Redundancy to encode information
• Digitizing quantum error

2.1 Quantum Error Correction Code (QECC)

Quantum error correction code is a mapping from $k$ logical qubits to $n$ physical qubits. Here, we must emphasize that $n$ is strictly greater than $k$, as it takes many physical qubits to realize one logical qubit. The idea is to use $n$ physical qubits to encode (protect) $k$ qubits of information. Exactly $n - k$ qubits are used for redundancy. This mapping can be shown as follows:

\[
|0_L\rangle = |000\rangle \quad (2)
\]
\[
|1_L\rangle = |111\rangle \quad (3)
\]

In the above example, $|0_L\rangle$ stands for the "logical" qubit, and it is realised by 3 physical qubits. Now suppose that with some small probability $p$, one of the physical qubits flipped, and we got $|001\rangle$. The original "logical" qubit can still be recovered, for example through a majority vote of qubits. We would conclude that the third qubit flipped, and the actual qubit was $|0_L\rangle$.

2.1.1 How to locate a bit flip?

Continuing the above example and representation, locating bit flips can be accomplished by looking at output sequences of a 2-qubit operator. These operators are $ZZI$ and $IZZ$, and each of the operators act on only one qubit in order. For example, $ZZI$ means a $Z$ gate is applied to both the first and the second qubit and the third qubit is left untouched. Recall that,

\[
Z|0\rangle = |0\rangle \quad (4)
\]
\[
Z|1\rangle = -|1\rangle \quad (5)
\]

Now, for a state $|\psi\rangle$ we can look at what the eigenvalues of these 2-qubit operators are. And if we apply both of these 2-qubit gates consecutively, we can determine which bit flipped. Now suppose that $|\psi\rangle = |100\rangle$. This means,

\[
ZZI|100\rangle = -|100\rangle \quad (6)
\]
\[
IZZ|100\rangle = |100\rangle \quad (7)
\]

The eigenvalues observed (in order) are (-1, +1). This sequence tells us that it’s the first qubit that is flipped. For instance, if the second qubit was flipped, we would instead observe a sequence that is (-1, -1). Similarly, we would see (+1, -1) if the third qubit was flipped. If one wishes to compute the phase flip of a qubit, then all $Z$ gates should be replaced by $X$ gates, and all $|0\rangle$ and $|1\rangle$ should be replaced by $|+\rangle$ and $|-\rangle$. This preserves the stabilizer
formalism, as the $X$ gate gives (+1, -1) as eigenvalues when it acts on $|+\rangle, |-\rangle)$. Everything else, just remains the same.

### 2.1.2 Check Matrix Formalism

The extension of how to locate bit flips to a more generalised case comes through the check matrix formalism. The idea of a check matrix is to create a set of qubit operations using the stabilizer formalism, with enough permutations sequences of eigenvalues to determine which qubit is flipped. Each row in the check matrix is a gate operation that needs to be applied to the system, and each column is representative of physical qubits. For example, the check matrix formalism for the above example would contain two rows, one for $IZZ$, and one for $ZZI$. It would also contain three columns, as there are three physical qubits in that system. The check matrix would be:

$$
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
\end{pmatrix}
$$

where 1’s stand for $Z$ (for bit flip) or $X$ (for phase flip) gates, and 0’s for the identity matrix. This matrix shows that first, $ZZI$ must applied, followed by $IZZ$. A more complicated example where 8 physical qubits are used would be

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\end{pmatrix}
$$

where we see that a series of 3 gate operations is necessary to encode enough sequences so that one can distinguish which qubit is flipped.

### 2.1.3 9-qubit Shor Code

Of course, using physical qubits to protect against bit flips would be no use, if one doesn’t also protect against phase flips, and vice versa. Unfortunately, each of the physical qubits individually need to also be protected by a second layer of concatenated physical qubits in this case. This gives rise to what is called the 9-qubit Shor Code, a 2-layer, 3x3 physical qubit set that protects against both phase and bit flips and encodes one logical qubit. This way, one layer protects against phase flips and the other against bit flips. The logical qubit encoded this way gives us:

$$
|0_L\rangle = \frac{1}{\sqrt{2}} |000\rangle + \frac{1}{\sqrt{2}} |111\rangle^{\otimes 3}
$$

$$
|1_L\rangle = \frac{1}{\sqrt{2}} |000\rangle - \frac{1}{\sqrt{2}} |111\rangle^{\otimes 3}
$$

In this case, operators to check whether phase flips or bit flips occured changes. We need 3 sets of bit flip checks, and 2 sets of phase flip checks. These gates are given below:
2.2 Projective Measurement

All above measurements gate operations (including examples of $ZZI$) should be done described below as a projective measurement. The circuit state at each time step is given below:

Figure 1: The circuit describing a projective measurement for operation $A$. Here, $A$ can mean any stabilizer n-qubit gate. For example, it can mean $IZZ$ described above. There needs to be an additional ancilla qubit for this process.
1. $|0\rangle |\psi\rangle$

2. $|+\rangle |\psi\rangle$

3. $\frac{1}{\sqrt{2}} (|0\rangle |\psi\rangle + |1\rangle A |\psi\rangle)$

4. $\frac{1}{2}[(|0\rangle + |1\rangle) |\psi\rangle + (|0\rangle - |1\rangle) A |\psi\rangle] = |0\rangle \frac{I + A}{2} |\psi\rangle + |1\rangle \frac{I - A}{2} |\psi\rangle$

where operators $\frac{I + A}{2}$ and $\frac{I - A}{2}$ are called "projectors". It can be shown that any arbitrary state $|\psi\rangle$ can be decomposed into orthogonal states as follows:

$$|\psi\rangle = \alpha |\psi_+\rangle + \beta |\psi_-\rangle$$

(17)

where $|\psi_+\rangle$ and $|\psi_-\rangle$ are the eigenstates of $A$ with eigenvalues (+1, -1) respectively. One can think of these states as "no error" and "error" states as well. Since when we have no error, stabilizer operators give us an eigenvalue of +1, and when we have error, it’s -1. Therefore, we can see further that:

$$\frac{I + A}{2} (\alpha |\psi_+\rangle + \beta |\psi_-\rangle) = \alpha \frac{1 + 1}{2} |\psi_+\rangle + \beta \frac{1 - 1}{2} |\psi_-\rangle = \alpha |\psi_+\rangle$$

(18)

which shows us that we recover the original "no error" state $|\psi_+\rangle$ with probability $\alpha$ and the "error" state $|\psi_-\rangle$ with probability $\beta$. So if $\alpha \sqrt{1 - \epsilon}$ and $\beta \sqrt{\epsilon}$ where $\epsilon \ll 1$, or in other word, where the error is small, we recover the "no error" state with high probability. This procedure shows that "projectors" actually project the arbitrary state $|\psi\rangle$ into one of the two states, "no error" and "error" states.

### 3 Surface Code

Showed in powerpoint.