CMSC 23700	**Introduction to Computer Graphics**	**Homework 1**
Winter 2012		**Assigned Jan 6**
		Due Noon Jan 13

Some of the main points from Jan 5 lecture can be summarized as:

$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}; \mathbf{b}_i \cdot \mathbf{b}_j = 1$ if $i = j$ else 0 — let $\mathcal{B}$ in an orthonormal basis

$$[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \cdot \mathbf{x} \\ \mathbf{b}_2 \cdot \mathbf{x} \\ \mathbf{b}_3 \cdot \mathbf{x} \end{bmatrix}$$ representation of $\mathbf{x}$ in $\mathcal{B}$

$\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3$decomposition of $\mathbf{x}$ along basis vectors

$[\mathbf{x}+\mathbf{y}]_\mathcal{B} = [\mathbf{x}]_\mathcal{B} + [\mathbf{y}]_\mathcal{B}$ — linearity of coordinate representation

$$[\mathbf{x}]_\mathcal{C} = \begin{bmatrix} \mathbf{b}_1 \cdot \mathbf{c}_1 & \mathbf{b}_2 \cdot \mathbf{c}_1 & \mathbf{b}_3 \cdot \mathbf{c}_1 \\ \mathbf{b}_1 \cdot \mathbf{c}_2 & \mathbf{b}_2 \cdot \mathbf{c}_2 & \mathbf{b}_3 \cdot \mathbf{c}_2 \\ \mathbf{b}_1 \cdot \mathbf{c}_3 & \mathbf{b}_2 \cdot \mathbf{c}_3 & \mathbf{b}_3 \cdot \mathbf{c}_3 \end{bmatrix} [\mathbf{x}]_\mathcal{B};$$ coord. transform between two orthonormal bases

$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}|\cos(\theta)$ — def. of dot product; θ is angle between $\mathbf{x}$ and $\mathbf{y}$

$|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ — vector length in terms of $\cdot$

$\mathbf{I}\mathbf{x} = \mathbf{x}$ — def. of identity tensor

$(\mathbf{x} \otimes \mathbf{y})\mathbf{v} = \mathbf{x}(\mathbf{y} \cdot \mathbf{v})$ — definition of tensor product $\otimes$

$\mathbf{x} \cdot \mathbf{y} = [\mathbf{x}]_\mathcal{B}^T [\mathbf{y}]_\mathcal{B}$ — computing dot products with components in $\mathcal{B}$

$[\mathbf{x} \otimes \mathbf{y}]_\mathcal{B} = [\mathbf{x}]_\mathcal{B} [\mathbf{y}]_\mathcal{B}^T$ — computing tensor products with components in $\mathcal{B}$

$$[\mathbf{T}]_\mathcal{B} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}; T_{ij} = \mathbf{b}_i \cdot \mathbf{T}\mathbf{b}_j$$ representation of $\mathbf{T}$ in $\mathcal{B}$

$\mathbf{T}_\text{scale} = \mathbf{I} + (\alpha - 1)(\hat{\mathbf{x}} \otimes \hat{\mathbf{x}})$ — scales by α along $\hat{\mathrm{x}}$

$\mathbf{T}_\text{shear} = \mathbf{I} + \beta\hat{\mathbf{x}} \otimes \hat{\mathbf{y}}$ — shears $\mathbf{v}$ along $\hat{\mathbf{x}}$ by $\beta\hat{\mathbf{y}} \cdot \mathbf{v}$

Feb 3: the LHS of "computing tensor products" was incorrectly $\mathbf{x} \otimes \mathbf{y}$ instead of $[\mathbf{x} \otimes \mathbf{y}]_\mathcal{B}$ in the original HW

1) Consider a 2-dimensional space with orthonormal basis $\mathcal{B} = \{\mathbf{x}, \mathbf{y}\}$.

1. What is $[\mathbf{I} + (\alpha - 1)\mathbf{x} \otimes \mathbf{x}]_\mathcal{B}$? (What are the components of this 2-by-2 matrix?)
2. Show that the same scaling transform could also have been written $\alpha\mathbf{x} \otimes \mathbf{x} + \mathbf{y} \otimes \mathbf{y}$.
3. What is a simpler way to write $\mathbf{x} \otimes \mathbf{x} + \mathbf{y} \otimes \mathbf{y}$? (hint: a single letter!) Why is this?
4. Let $\mathbf{v} = \mathbf{x} + \mathbf{y}$ and $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$. What is $[\hat{\mathbf{v}}]_\mathcal{B}$? (That is, what are the components of this 2-vector?)
5. What is $[\mathbf{I} + (\alpha - 1)\hat{\mathbf{v}} \otimes \hat{\mathbf{v}}]_\mathcal{B}$?

2) Consider a 3-dimensional space with orthonormal basis $\mathcal{B} = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$.

1. Describe geometrically what the transform $\mathbf{T}_1 = \mathbf{I} + \mathbf{z} \otimes (\alpha\mathbf{x} + \beta\mathbf{y})$ does.
2. What is $[\mathbf{T}_1]_\mathcal{B}$? (What will this 3x3 matrix look like?)

3. Let $\mathbf{T}_2 = \mathbf{I} + \alpha \mathbf{y} \otimes \mathbf{v}$, for an arbitrary vector $\mathbf{v}$. What is $[\mathbf{T}_2]_{\mathcal{B}}$? The answer should involve the v_i components of $[\mathbf{v}]_{\mathcal{B}}$.

3) The Real-Time Rendering book (Table 1.2 page 7) uses the symbol $\otimes$ to represent component-wise multiplication of two vectors $\mathbf{u} = (u_1 \;\; u_2)^T$ and $\mathbf{v} = (v_1 \;\; v_2)^T$ as

$$\mathbf{u}\text{“}\otimes\text{”}\mathbf{v} = (u_1 v_1 \;\; u_2 v_2)^T, \tag{1}$$

where we're using two-dimensional vectors for simplicity. The book doesn't follow the same practice as above in notationally distinguising between geometric vectors and their representation in some basis, but given some orthonormal basis $\mathcal{B}$ we could interpret this "$\otimes$" as a multiplication of two geometric vectors $\mathbf{u}$ and $\mathbf{v}$ that gives back a new geometric vector $\mathbf{u}\text{“}\otimes\text{”}\mathbf{v}$, with a product that happens to depend on the components of $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{B}$:

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow [\mathbf{u}\text{“}\otimes\text{”}\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} u_1 v_1 \\ u_2 v_2 \end{bmatrix} \tag{2}$$

$$\Rightarrow \mathbf{u}\text{“}\otimes\text{”}\mathbf{v} = u_1 v_1 \mathbf{b}_1 + u_2 v_2 \mathbf{b}_2 \tag{3}$$

If you had chosen a different orthonormal basis, would the vector $\mathbf{u}\text{“}\otimes\text{”}\mathbf{v}$ always be the same? Find a two-dimensional counter-example: two orthonormal bases $\mathcal{B}$ and $\mathcal{C}$, and two vectors $\mathbf{u}$ and $\mathbf{v}$, so that the product $\mathbf{u}\text{“}\otimes\text{”}\mathbf{v}$ can produce two very different (coordinate-free, geometric) answers, depending on the basis used. You can draw a picture, or symbolically define $\mathcal{C}$ in terms of $\mathcal{B}$, and the two vectors in terms of $\mathcal{B}$. This is a question that take much less space answer, than to ask, and don't over-think it!