CMSC 23700	Introduction to Computer Graphics	Homework 2
Winter 2012		Assigned Jan 26
		Due Noon Feb 7

1) "Deriving View and Projection Transforms" (`handout-02-projections.pdf`) derived the perspective transformation matrix in its equation (25), which can be written in a simplified form as

$$M_{\text{pers}} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{1}$$

We know that the bottom row of this matrix, followed by a normalization of the homogeneous clip coordinates, creates a division by the third view-space coordinate. The mapping of the third view-space coordinate to its normalized device coordinate is then:

$$n \longrightarrow \frac{cn + d}{n} \tag{2}$$

This is a *non-linear* function of n, yet lines in world space appear as lines on the screen. Hmmm.

1. Show that (2) is not a linear function of n, that is, a function $f(x)$ for which $f(x + y) = f(x) + f(y)$ and $f(\alpha x) = \alpha f(x)$. It suffices to supply a single counter example (for general c and d).
2. Consider a general line, parameterized by

   $$\mathbf{r}(s) = \mathbf{p} + s\mathbf{v} \tag{3}$$

   where s is any real number and $|\mathbf{v}| = 1$. The line goes through point $\mathbf{p}$ (at $s = 0$) along direction $\mathbf{v}$. Let the view-space coordinates of these be notated as $[\mathbf{p}]_{\mathcal{V}} = [p_1 \;\; p_2 \;\; p_3]^T$ and $[\mathbf{v}]_{\mathcal{V}} = [v_1 \;\; v_2 \;\; v_3]^T$. Express (3) in *homogeneous coordinates*.
3. Write out the result of mapping the homogenous coordinates of $\mathbf{r}(s)$ through M_{pers}, followed by homogeneous coordinate normalization (dividing by the fourth coordinate).
4. Show that the expression for the previous part traces out a line in normalized device coordinates. Hint: one way of doing this is to find the derivative with respect to s, and show that the direction (though not the length) of the derivative is independent of s, which means the path is always going in the same direction. This means you can neglect scaling terms that appear on all three normalized device coordinates. More clever approaches are welcome.
5. Argue (without resorting to formulae) that given the above result, lines in *world*-space must indeed map to lines in the final rendered image in screen-space. Hint: what are the other transforms involved besides the projection transform?

2) Affine transformations can be represented by 4×4 homogeneous matrices of the following form:

$$\begin{bmatrix} \mathbf{M} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

where $\mathbf{M}$ is a 3×3 matrix and $\mathbf{t}$ is a vector. We can use $\langle \mathbf{M} \mid \mathbf{t} \rangle$ as a more compact notation for this class of matrices. Applying the transformation to a homogeneous point is

$$\langle \mathbf{M} \mid \mathbf{t} \rangle \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} = \mathbf{M}\mathbf{v} + \mathbf{t}$$

If we restrict ourselves to uniform scaling, rotation, and translation, then these matrices have the form $\langle s\mathbf{R} \mid \mathbf{t} \rangle$, where s is a scalar and $\mathbf{R}$ is a rotation matrix. Let $\mathbf{I}$ be the identity matrix and $\mathbf{0}$ the zero vector. Using this notation:

1. Express $\langle \mathbf{M}_1 \mid \mathbf{t}_1 \rangle \langle \mathbf{M}_2 \mid \mathbf{t}_2 \rangle$ as a single affine transform.
2. Factor $\langle s_0 \mathbf{R}_0 \mid \mathbf{t}_0 \rangle$ into three affine transforms, with only one of s_0, $\mathbf{R}_0$ and $\mathbf{t}_0$ showing up in each term (that is, find an equivalent ordering of the scaling, translation, and rotation in $\langle s_0 \mathbf{R}_0 \mid \mathbf{t}_0 \rangle$).
3. Expand and simplify $\langle s\mathbf{R} \mid \mathbf{t} \rangle \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}$.
4. Expand and simplify $\langle s_1 \mathbf{R}_1 \mid \mathbf{t}_1 \rangle \langle s_2 \mathbf{R}_2 \mid \mathbf{t}_2 \rangle$.
5. Find an expression for $\langle s\mathbf{R} \mid \mathbf{t} \rangle^{-1}$ (knowing that $\langle s\mathbf{R} \mid \mathbf{t} \rangle^{-1} \langle s\mathbf{R} \mid \mathbf{t} \rangle = \mathbf{I}$), and then express this inverse as a product of three transforms as in part 2 above.

Show the steps of your derivations.