An self-contained explanation of image orientation and the "measurement frame", with connections to the NRRD format (Version 0.5)

Gordon L. Kindlmann, University of Chicago, glk@uchicago.edu

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1 Background and Notation

This document aims to be a self-contained explanation of raster image orientation, and the notion of "measurement frame", as supported by the NRRD image format. The measurement frame is used to disambiguate the interpretation of vector and tensor values that may be stored as sample values inside a NRRD. This is a more careful and more mathematical description of information that should be in `http://teem.sourceforge.net/nrrd/format.html`. Section 3 may be useful for understanding diffusion-weighted MRI volumes, or diffusion tensor volumes, in NRRD files that use the "`measurement frame:`" field, as described at `http://www.na-mic.org/Wiki/index.php/NAMIC_Wiki:DTI:Nrrd_format`.

Following conventions of tensor analysis, we are careful to distinguish between vectors and tensors as coordinate-free objects (e.g. $\mathbf{v}$, in bold-face), and the representation of the vectors or tensors in some particular coordinate system (e.g. $[\mathbf{v}]_\mathcal{B} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, with v_i in non-bold italics). NRRD is general with respect to dimensionality, both for the dimension of the image data, and for the world-space in which its orientation is defined, but it has some specific semantics for three-dimensional world-space, which is all that we describe here. Bases are notated in calligraphic font, e.g. $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. In this context it is sufficient to assume that all bases are orthonormal.

It is important to be able to simultaneously consider coordinate-free (direct notation) and coordinate-based (matrix notation) expressions of the same quantities. In basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, a (coordinate-free) vector $\mathbf{v}$ has coordinates (or coefficients) v_i, and tensor $\mathbf{D}$ has coefficients D_{ij}. Coefficients are found by dot products with basis vectors.

$$\text{direct notation} \longleftrightarrow \text{matrix notation} \tag{1}$$

$$\mathbf{v} = \sum_i v_i \mathbf{b}_i \longleftrightarrow [\mathbf{v}]_\mathcal{B} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix},\ v_i = \mathbf{v} \cdot \mathbf{b}_i \tag{2}$$

$$\mathbf{D} = \sum_{ij} D_{ij} \mathbf{b}_i \otimes \mathbf{b}_j \longleftrightarrow [\mathbf{D}]_\mathcal{B} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix},\ D_{ij} = \mathbf{b}_i \cdot \mathbf{D}\mathbf{b}_j \tag{3}$$

2 Image Orientation

2.1 World-space and its bases

We call the continuous three-dimensional space in which an image has been sampled *world space*. Different research areas have different ideas about how best to identify that three-dimensional space. In neuroimaging, it is common to define that space with respect to the human who's brain is being scanned, so a 3-D basis is defined along the anatomical axes of anterior-posterior, inferior-superior, and left-right. Some groups prefer an orthonormal basis defined as

$$\mathcal{B}_{\text{LPS}} = \left\{ \begin{matrix} \text{from right towards left} \\ \text{from anterior towards posterior} \\ \text{from inferior towards superior} \end{matrix} \right\}. \tag{4}$$

Others prefer

$$\mathcal{B}_{\text{RAS}} = \left\{ \begin{matrix} \text{from left towards right} \\ \text{from posterior towards anterior} \\ \text{from inferior towards superior} \end{matrix} \right\}. \tag{5}$$

Either basis is equally useful or logical, but preference for one or the other can be strong. There are also established radiological conventions for how slices of brain images should be orientated on a 2-dimensional display, but these conventions are unrelated to the representation of 3-D image orientation.

2.2 Index-space

Let $F[i, j, k]$ be a 3-D image data volume indexed by integers i, j, k. Defining image orientation does not require us to define the semantics of the ordering of indices ($F[i, j, k]$ versus $F[k, j, i]$), but this is usually related to the raster ordering of axes on disk or in memory. The space of indices (i, j, k) into $F[]$ is termed *index space*.

Even though the (i, j, k) indices in principle form a vector space, it is treated differently than world space. It is not a continuous space because we expect the indices to be non-negative integers, and it makes much less sense to consider diverse bases for index space, in which a given index vector might have different representations, as compared to the sensibility of simultaneously considering different bases for world-space (like $\mathcal{B}_{\text{RAS}}$ and $\mathcal{B}_{\text{LPS}}$).

As a result, we don't maintain the same discipline in distinguishing between index space vectors and their representation in a basis, as we do with world space vectors ($\mathbf{v} \neq [\mathbf{v}]_{\mathcal{B}}$). The index space basis is implicit, and a point in index space is simply identified with its integral coordinates (i, j, k).

2.3 Definition of image orientation

Image orientation is an affine transform from an index space vector (i, j, k) to world space vector $\mathbf{x}$, combining a linear transform $\mathbf{A}$ with translation $\mathbf{t}$. $\mathbf{A}$ carries all the information about axis scaling and rotation, and the $\mathbf{t}$ carries the information about where the image is located, in terms of the location of its $(0,0,0)$ sample.

$$\begin{bmatrix} \\ \mathbf{x} \\ \\ \end{bmatrix}_{\mathcal{B}} = \left[\begin{bmatrix} & & \\ & \mathbf{A} & \\ & & \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} \\ \mathbf{t} \\ \\ \end{bmatrix}_{\mathcal{B}} \right] \begin{bmatrix} i \\ j \\ k \\ 1 \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & t_1 \\ A_{21} & A_{22} & A_{33} & t_2 \\ A_{31} & A_{32} & A_{33} & t_3 \end{bmatrix} \begin{bmatrix} i \\ j \\ k \\ 1 \end{bmatrix} \tag{7}$$

Note the use of homogeneous coordinates (suffixing the index space vector with 1), which allows translation by $\mathbf{t}$ to be represented by a matrix multiplication. The A_{ij} coefficients are naturally organized by the columns of $[\mathbf{A}]_{\mathcal{B}}$:

$$[\mathbf{A}]_{\mathcal{B}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{33} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \left[\begin{bmatrix} \\ \mathbf{a}_1 \\ \\ \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} \\ \mathbf{a}_2 \\ \\ \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} \\ \mathbf{a}_3 \\ \\ \end{bmatrix}_{\mathcal{B}} \right] \tag{8}$$

The vectors $\mathbf{a}_i$ have the following geometric meaning, independent of any world-space basis. The difference between the spatial locations of $F[1,0,0]$ and $F[0,0,0]$ is given by the vector $\mathbf{a}_1$. The spatial vector from $F[0,0,0]$ to $F[0,1,0]$ and to $F[0,0,1]$ are given by $\mathbf{a}_2$ and $\mathbf{a}_3$, respectively. Since the image sampling grid is regular, we can say

$$\text{world-space-location}(F[i,j,k]) = \mathbf{t} + i\mathbf{a}_1 + j\mathbf{a}_2 + k\mathbf{a}_3. \tag{9}$$

The per-axis spacings between samples are the vector lengths $|\mathbf{a}_1|$, $|\mathbf{a}_2|$, and $|\mathbf{a}_3|$.

Given a different basis for world-space (say, $\mathcal{B}_{\text{RAS}}$ versus $\mathcal{B}_{\text{LPS}}$), the concrete representation of image orientation (in terms of $[\mathbf{A}]_{\mathcal{B}}$ and $[\mathbf{t}]_{\mathcal{B}}$) will be different, even though in principle the image orientation itself (in terms of $\mathbf{A}$ and $\mathbf{t}$) is the same.

2.4 Image orientation in NRRD

The NRRD file format fields starting with "`space`" define image orientation. First, the world space basis $\mathcal{B}$ has to be communicated. Various commonly-used spaces have been given explicit named, which a NRRD file can refer to with the "`space:`" field. For neuroimaging that isn't time-dependent, the most common bases basis $\mathcal{B}$ are all three-dimensional:

- "`space: RAS`" or "`space: right-anterior-superior`": For medical data, a patient-based right-handed basis, with ordered basis vectors pointing towards right, anterior, and superior, respectively. This space is used in the NIFTI-1 extension to the Analyze format.

- "`space: LAS`" or "`space: left-anterior-superior`": For medical data, a patient-based left-handed basis, with ordered basis vectors pointing towards left, anterior, and superior, respectively. This space is used in the Analyze 7.5 format.

- "`space: LPS`" or "`space: left-posterior-superior`": For medical data, a patient-based right-handed basis, with ordered basis vectors pointing towards left, posterior, and superior, respectively. This space is used in DICOM (version 3 onwards).

See `http://teem.sourceforge.net/nrrd/format.html#space` for the other available named spaces. If there is not a NRRD-named space that corresponds to the intended basis $\mathcal{B}$ in which to quantify image orientation, one simply identifies its integral dimension n with

$$\texttt{space dimension:}n$$

In any case, the basis $\mathcal{B}$ in which image orientation is defined (in terms of the coefficients $[\mathbf{A}]_\mathcal{B}$ and $[\mathbf{t}]_\mathcal{B}$), is never itself defined relative to some other more fundamental basis. This is analogous to how $\mathcal{B}_{\text{LPS}}$ in (4) or $\mathcal{B}_{\text{RAS}}$ in (5) were not defined in terms of some other set of coordinate vectors.

NRRD organizes image orientation, as defined by the $\mathbf{A}$ and $\mathbf{t}$ of (6) and (7), into per-array and per-axis information. The translation $\mathbf{t}$ (image location) is per-array information, given by:

$$\texttt{space origin: (}t_1\texttt{,}t_2\texttt{,}t_2\texttt{)}$$

The linear transform $\mathbf{A}$ (image axis scaling and rotation) is decomposed into per-axis information, given by:

$$\texttt{space directions: (}A_{11}\texttt{,}A_{21}\texttt{,}A_{31}\texttt{), (}A_{12}\texttt{,}A_{22}\texttt{,}A_{32}\texttt{), (}A_{13}\texttt{,}A_{23}\texttt{,}A_{33}\texttt{)}$$

The three vectors defined by this are the column vectors of $\mathbf{A}_\mathcal{B}$, which are also the components of the $[\mathbf{a}_i]_\mathcal{B}$ vectors used in (9). Whenever listing per-axis information, NRRD chooses the fast to slow ordering. The "fastest" axis of index space is the one whose coordinate changes most rapidly with a linear traversal of samples in memory or on disk.

3 Measurement Frames in NRRD

So far, we have considered the orientation of the imaging grid relative to world space, but we have not considered anything about the values stored in the image. If they are scalars, then no other coordinate frame needs defining. However, if the stored values are the components of vectors or tensors, then they must have been measured with respect to some basis $\mathcal{M} = \{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$, via:

$$[\mathbf{v}]_\mathcal{M} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}; v_i = \mathbf{v} \cdot \mathbf{m}_i \tag{10}$$

$$[\mathbf{D}]_\mathcal{M} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix}; D_{ij} = \mathbf{m}_i \cdot \mathbf{D}\mathbf{m}_j \tag{11}$$

To meaningfully interpret the stored v_i or D_{ij} components, we need to know the basis $\mathcal{M}$.

Note that in the case of using NRRD to store diffusion-weighted image (DWI) volumes, it is also helpful to indicate the basis $\mathcal{M}$, even though DWIs are scalars, because the diffusion-sensitizing gradient vectors $\mathbf{g}_i$ are stored in the NRRD meta-data. The diffusion-sensitizing gradients are expressed there as $[\mathbf{g}_i]_\mathcal{M}$, using some measurement basis $\mathcal{M}$.

In NRRD, the $\mathcal{M}$ basis is communicated in terms of the world-space basis $\mathcal{B}$ that was used to represent image orientation (Section 2.4). Specifically, we record the coefficients of the matrix $T_{\mathcal{BM}}$ that, for an arbitrary vector $\mathbf{v}$, transforms $[\mathbf{v}]_{\mathcal{M}}$ to $[\mathbf{v}]_{\mathcal{B}}$.

$$[\mathbf{v}]_{\mathcal{B}} = T_{\mathcal{B}M}[\mathbf{v}]_{\mathcal{M}} \tag{12}$$

$$\begin{bmatrix} \mathbf{v}\cdot\mathbf{b}_1 \\ \mathbf{v}\cdot\mathbf{b}_2 \\ \mathbf{v}\cdot\mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} \mathbf{v}\cdot\mathbf{m}_1 \\ \mathbf{v}\cdot\mathbf{m}_2 \\ \mathbf{v}\cdot\mathbf{m}_3 \end{bmatrix} \tag{13}$$

This matrix is stored in the "`measurement frame:`" field:

`measurement frame:` (T_{11}, T_{21}, T_{31}), (T_{12}, T_{22}, T_{32}), (T_{13}, T_{23}, T_{33})

This is in analogy to the way that the linear transform $\mathbf{A}$ in image orientation is stored in terms of the column vectors of $[\mathbf{A}]_{\mathcal{B}}$, the matrix that transforms index-space vectors into world-space (minus the image orientation translation).

This implies how to transform $[\mathbf{D}]_{\mathcal{M}}$, a tensor $\mathbf{D}$ measured in basis $\mathcal{M}$ (and stored in the NRRD file), to $[\mathbf{D}]_{\mathcal{B}}$, the same tensor measured in world-space basis $\mathcal{B}$. We start by recognizing that the tensor acts as a linear transform on world-space, from $\mathbf{u}$ to $\mathbf{v} = \mathbf{D}\mathbf{u}$:

$$\mathbf{v} = \mathbf{D}\mathbf{u} \tag{14}$$

$$\Rightarrow [\mathbf{v}]_{\mathcal{B}} = [\mathbf{D}]_{\mathcal{B}}\,[\mathbf{u}]_{\mathcal{B}} \tag{15}$$

$$\Rightarrow T_{\mathcal{BM}}[\mathbf{v}]_{\mathcal{M}} = [\mathbf{D}]_{\mathcal{B}}\,T_{\mathcal{BM}}[\mathbf{u}]_{\mathcal{M}} \tag{16}$$

$$\Rightarrow [\mathbf{v}]_{\mathcal{M}} = T^{-1}_{\mathcal{BM}}[\mathbf{D}]_{\mathcal{B}}\,T_{\mathcal{BM}}[\mathbf{u}]_{\mathcal{M}} \tag{17}$$

$$\Rightarrow [\mathbf{D}]_{\mathcal{M}} = T^{-1}_{\mathcal{BM}}[\mathbf{D}]_{\mathcal{B}}\,T_{\mathcal{BM}} \tag{18}$$

$$\Rightarrow [\mathbf{D}]_{\mathcal{B}} = T_{\mathcal{BM}}[\mathbf{D}]_{\mathcal{M}}T^{-1}_{\mathcal{BM}} \tag{19}$$

The last line gives the transform from $[\mathbf{D}]_{\mathcal{M}}$ to $[\mathbf{D}]_{\mathcal{B}}$, in terms of the measurement frame matrix $T_{\mathcal{BM}}$ and its inverse (compare to (12), for vectors). If both bases $\mathcal{B}$ and $\mathcal{M}$ are orthonormal, $T^{-1}_{\mathcal{BM}} = T^{\mathsf{T}}_{\mathcal{BM}}$.

For completeness, we can also describe the measurement frame matrix $T_{\mathcal{BM}}$ in terms of pair-wise dot products between elements of $\mathcal{B}$ and elements of $\mathcal{M}$.

$$[\mathbf{v}]_{\mathcal{B}} = T_{\mathcal{B}M}[\mathbf{v}]_{\mathcal{M}} \tag{20}$$

$$\Rightarrow [\mathbf{m}_1]_{\mathcal{B}} = \begin{bmatrix} \mathbf{m}_1\cdot\mathbf{b}_1 \\ \mathbf{m}_1\cdot\mathbf{b}_2 \\ \mathbf{m}_1\cdot\mathbf{b}_3 \end{bmatrix} = T_{\mathcal{B}M}[\mathbf{m}_1]_{\mathcal{M}} = T_{\mathcal{B}M}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{21} \\ T_{31} \end{bmatrix}. \tag{21}$$

Similar expressions for $[\mathbf{m}_2]_{\mathcal{B}}$ and $[\mathbf{m}_3]_{\mathcal{B}}$ lead to

$$T_{\mathcal{B}M} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1\cdot\mathbf{m}_1 & \mathbf{b}_1\cdot\mathbf{m}_2 & \mathbf{b}_1\cdot\mathbf{m}_3 \\ \mathbf{b}_2\cdot\mathbf{m}_1 & \mathbf{b}_2\cdot\mathbf{m}_2 & \mathbf{b}_2\cdot\mathbf{m}_3 \\ \mathbf{b}_3\cdot\mathbf{m}_1 & \mathbf{b}_3\cdot\mathbf{m}_2 & \mathbf{b}_3\cdot\mathbf{m}_3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \\ \mathbf{m}_1 \\ \\ \end{bmatrix}_{\mathcal{B}} & \begin{bmatrix} \\ \mathbf{m}_2 \\ \\ \end{bmatrix}_{\mathcal{B}} & \begin{bmatrix} \\ \mathbf{m}_3 \\ \\ \end{bmatrix}_{\mathcal{B}} \end{bmatrix}. \tag{22}$$

So, the column vectors of $T_{\mathcal{B}M}$ are the representations, in world-space basis $\mathcal{B}$, of the $\mathbf{m}_i$ vectors that make up the measurement basis $\mathcal{M}$.