Abstract Algebra Math 258 Practice Problems

1. All homework problems
2. All midterm problems
3. Prove that any finite integral domain is a field.
4. Let R be a ring with 1. Is the set $\{(r,r)|r \in R\}$ (a) a subring? (b) an ideal?
5. Let R be a ring with 1 and I an ideal of R. Prove that R/I is commutative iff rs-sr is an element of I for all r,s, $\in R$.
6. Prove that if F is a field then $F[x]$ is a PID.
7. Let V be a finitely generated vector space. (a) Prove that any linearly independent set in V can be completed to a basis. (b) Prove that any two bases of V have the same size.
8. Compute the determinant of the following matrix with entries in $\mathbb{Z}[x]$:

$$\begin{pmatrix} 3x & 4x+2 & 3 \\ 5x^2+x+1 & 4x^2+2 & x \\ -5x^2+5x-1 & -4x^2+8+2 & 6-x \end{pmatrix}$$

9. (a) What is the ring of fractions for $2\mathbb{Z} \subset \mathbb{Z}$?
(b) What is the ring of fractions for $2\mathbb{Z} \subset \mathbb{Q}$?
(c) What is the ring of fractions for $2\mathbb{Z} \subset \mathbb{R}$?
10. Give an example of a non-trivial ring that is not a Euclidean Domain. Justify your answer!
11. What is the gcd of $f(x) = x^5 + 1$ and $x^3 + 1$ in $\mathbb{Q}[x]$?
12. (I will not ask you this kind of question on the final exam , but it is good practice.)

    Let A be the matrix:

$$\begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$$

Compute the following:
(a) eigenvalues
(b) eigenspaces
(c) characteristic polynomial
(d) minimal polynomial
(e) rank
(f) determinant
(g) inverse
(h) Jordan canonical form

13. Page 344 problems 19 and 20.

14. Prove the Module isomorphism theorems (or give a convincing sketch).

15. Page 350 Problem 9.

16. Page 356 Problem 9.

17. Let R be a PID and M an R-module. (a) Define the torsion submodule of M $\mathrm{Tor}(M)$. (b) For a submodule $N \subseteq M$ define the annihilator of N $\mathrm{Ann}(N)$. (c) If M is a cyclic R-module, prove that $C \equiv R/\mathrm{Ann}(C)$

18. Let V be a finite dimensional vector space over F and T a linear transformation from V to itself. Consider V as an $F[x]$-module with action given by T.

    The fundamental theorem of finitely generated modules over a PID gives a decomposition of V into direct sums of quotients of $F[x]$. (a) State this decomposition (I'm looking for something like equation 12.1 in the text, you could also state the general theorem). (b) In order to form the rational canonical form of T we picked a particularly nice basis for each summand. State this basis. (c) Give the matrix representation for T on this summand with respect to this basis. (c) Give the matrix representation for T on V with respect to the union of the bases (i.e the Rational canonical form for T).

19. Let M be a block diagonal matrix:

$$\begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & A_k \end{pmatrix}$$

Prove that the characteristic polynomial of M is the product of the characteristic polynomials of the A_is.

20. Let V be a finite dimensional vector space over F and T be a linear transformation from V to V. Assume F contains all eigenvalues of T. Consider V as an $F[x]$-module with respect to T. (a) What form do the elementary divisors have? (b) The fundamental theorem of finitely generated modules over a PID gives a decomposition of V in terms of cyclic $F[x]$-modules. State this decomposition. (c) To form the Jordan canonical form we picked a particularly nice basis for each summand. State this basis. (d) What is the matrix representation of T on this summand with respect to this basis? (e) Give the matrix representation of T with respect to the union of these bases. (ie the Jordan canonical form).