Homework 5

1. Recall that the twisted cubic was defined as $\mathbf{V}(y - x^2, z - x^3)$.

   For any $f \in \mathbb{R}[x, y, z]$ show that f can be written as:

$$h_1(y - x^2) + h_2(z - x^3) + r$$

   where $h_1, h_2 \in \mathbb{R}[x, y, z]$ and r is a polynomial in x alone.

2. (a) Show that the twisted cubic can be parameterized as $\{(t, t^2, t^3) : t \in \mathbb{R}\}$.

   (b) Prove that $I(V(y - x^2, z - x^3)) =< y - x^2, z - x^3 >$.

   (Hint, use part (a) and problem 1)

3. Prove that $I(V(x^n, y^m)) =< x, y >$ for all $n, m \in \mathbb{Z}^+$.

4. Prove the following strengthing of Dickson's Lemma. If I is a monomial ideal $I =< x^\alpha : \alpha \in A >$, then $I =< x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_s} >$. where $\alpha_1, \alpha_2, \ldots, \alpha_s \in A$. Namely the finite generating set may be formed from elements of A. (You may use the Lemma we proved in class, that every monomial ideal has a finite generating set.)

5. Let $>$ be a relation on $\mathbb{Z}_{\geq 0}$ such that

   (i) $>$ is a total ordering on $\mathbb{Z}_{\geq 0}$.

   (ii) If $\alpha > \beta$ and $\gamma \in \mathbb{Z}_{\geq 0}$ then $\alpha + \gamma > \beta + \gamma$.

   then $>$ is a well-ordering iff $\alpha \geq 0 \;\; \forall \alpha \in \mathbb{Z}_{\geq 0}$.

   (Hint: Use Dickson's Lemma)