# Abstract Algebra Math 259 Practice Problems

1. State and prove the Hilbert Basis Theorem. (You may assume Dickson's Lemma).

2. Fix a monomial ordering on $k[x_1, \ldots, x_n]$ and let $G = \{g_1, \ldots, g_m\}$ be a Gröbner basis for a non-zero ideal $I$ in $k[x_1, \ldots, x_n]$.

   Prove that a polynomial $f \in k[x_1, \ldots, x_n]$ has a unique remainder when divided by $G$.

3. (a) Prove that $f(x) = x^3 + x + 1$ is irreducible over $\mathbb{Z}_2$.

   (b) Let $E$ be the smallest extension field of $\mathbb{Z}_2$ containing a solution $\alpha$ of $f(x)$. What is the degree of $\alpha$ over $\mathbb{Z}_2$?

   (c) Give a basis for $E$ as a vector space over $\mathbb{Z}_2$.

   (d) Write the element $\alpha^3$ in terms of your basis.

4. (a) Prove that in general the union of two fields may not be a field.

   (b) Prove that $\cup_{i=1}^{\infty} F_i$ is a field if the $F_i$ are fields satisfying:

   $$F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$$

   (c) Give an example of an infinite algebraic extension.

5. Let $f_1 = xy^2$ and $f_2 = x + y$ be polynomials in $k[x, y]$ where $k$ is algebraically closed. Using lexicographic order and $x > y$,

   (a) Compute a reduced Gröbner basis for $\langle f_1, f_2 \rangle$.

   (b) Prove whether or not $f = x^2 + 2xy + y^3 + y^2$ is in $\langle f_1, f_2 \rangle$.

   (c) Prove whether or not $\langle f_1, f_2 \rangle$ is a radical ideal.

   (d) Prove whether or not $\mathbf{V}(\langle f_1, f_2, \rangle) = \emptyset$.

6. Let $K$ be the splitting field for the polynomial $x^4 - 4$ over $\mathbb{Q}$.

   (a) Find $K$.

   (b) What is the degree $[K : \mathbb{Q}]$?

7. Let $K$ be the splitting field for the polynomial $x^4 - 2$ over $\mathbb{Q}$.

   (a) Find $K$.

   (b) What is the degree $[K : \mathbb{Q}]$?

8. (a) Prove that $\mathbb{Q}(\sqrt{2})$ is a Galois extension over $\mathbb{Q}$

   (b) Prove that $\mathbb{Q}$ adjoin the 4th root of 2 is Galois over $\mathbb{Q}(\sqrt{2})$

   (c) Prove that $\mathbb{Q}$ adjoin the 4th root of 2 is not Galois over $\mathbb{Q}$.

9. (a) Let $K - F$ be any finite extension. Prove that $K - F$ is Galois if and only if $F$ is the fixed field of $\mathrm{Aut}(K - F)$.

   (b) Let $G$ be any finite subgroup of the automorphisms of a field $K$ and let $F$ be the corresponding fixed field. Prove that $K - F$ is Galois with Galois group $G$.

10. Let $E$ be an extension field of $F$ and $\alpha \in E$. Prove that $\alpha$ is transcendental over $F$ if and only if the evaluation homomorphism $\phi_\alpha$ gives an isomorphism of $F[x]$ with a subring of $E$.

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12. Prove that the degree of a splitting field of a polynomial of degree $n$ over $F$ is at most $n!$.

13. For a fixed monomial ordering, prove that an ideal has a unique reduced Gröbner basis.

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