Midterm 2 - Math 201

Show all of your work. Justify all of your answers.

Problem 1 Let F be the vector field: $(x+y)\mathbf{i} + y^2\mathbf{j} + (x-y)\mathbf{k}$.
(a) Find curl F. curl $F = -1\mathbf{i} - 1\mathbf{j} - 1\mathbf{k}$
(b) Find div F. div $F = 1 + 2y$
(c) Find grad(div F). grad(div F) $= 2\mathbf{j}$
(d) Find div(curl F). div(curl F) $= 0$. This is always 0!

Problem 2 State the integral formulas of the three integral theorems: Green's theorem, Gauss (divergence) theorem, and Stoke's theorem. In each formula, state what your terms are. For example, if you have an $\mathbf{n}$ in your formula, say what $\mathbf{n}$ is.

Problem 3 Let F be the vector field: $2x\mathbf{i}+2\mathbf{j}+z\mathbf{k}$. Verify the divergence theorem for this vector field over the cube with corners: $(0,0,0), (0,0,2), (0,2,0), (2,0,0), (0,2,2), (2,0,2), (2,2,0), (2,2,2)$ (each edge has length 2).

The divergence theorem states that:

$$\int\int_S F \cdot \mathbf{n} dS = \int\int\int_V divF dv$$

$divF = 2+0+1 = 3$. So the right hand side is $3\int\int\int_V dv$. The volume of this cube is 8. Therefore we get 24 for the right side.

To compute the left side, we break up the surface of the cube into its 6 faces. These faces are: $x=0$, $x=2$, $y=0$, $y=2$, $z=0$, $z=2$.
The normal to $x=0$ is $-\mathbf{i}$, $F \cdot \mathbf{n} = -2x$, but $x=0$ so this face does not contribute to the surface integral.
The normal to $x=2$ is $\mathbf{i}$, $F \cdot \mathbf{n} = 2x$, but $x=2$ so we have $4\int\int_S dS = 16$ (since the area of any side of the cube is 4).
The normal to $y=0$ is $-\mathbf{j}$, $F \cdot \mathbf{n} = -2$, $-2\int\int_S dS = -8$.
The normal to $y=2$ is $\mathbf{j}$, $F \cdot \mathbf{n} = 2$, $2\int\int_S dS = 8$.
The normal to $z=0$ is $-\mathbf{k}$, $F \cdot \mathbf{n} = -z$, but $z=0$ so this face does not contribute to the surface integral.
The normal to $z=2$ is $\mathbf{k}$, $F \cdot \mathbf{n} = z$, but $z=2$ so we have $2\int\int_S dS = 8$.
Therefore $\int\int_S F \cdot \mathbf{n} dS = 16 - 8 + 8 + 8 = 24$.

Problem 4 Let F be the vector field: $(-y\mathbf{i} + x\mathbf{j})/(x^2+y^2)$. Evaluate the

path integral:$\oint F \cdot d\mathbf{r}$ counterclockwise around a unit circle centered at the origin.
Let $x = \cos\, u$ and $y = \sin\, u$.
Then $F =$ - $\sin u\mathbf{i}$ + $\cos u\mathbf{j}$.
$\frac{d\mathbf{r}}{du}$ = -$\sin u\mathbf{i}$ + $\cos u\mathbf{j}$.
$F \cdot \frac{d\mathbf{r}}{du} = \sin^2 u + \cos^2 u$.
Therfore we are left with: $\int_0^{2\pi} du = 2\pi$.