CMSC-37110 Discrete Mathematics
THIRD MIDTERM EXAM December 1, 2005

This is an 80-minute exam; it contributes **26%** to your course grade.

Do **not** *use book or notes.* You **may** use a calculator for *basic arithmetic* but not for more advanced functions such as binomial coefficients or determinants. **Show all your work. A mere statement of the answer generally does not earn partial credit.** If you are not sure of the meaning of a problem or the amount of detail required, **ask the proctor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (4 points) Let $\{b_n\}$ be a sequence of positive numbers such that $b_{n+1} = O(b_n)$. Prove: $\ln(b_n) = O(n)$.

2. (5 points) True or false: $\pi(x) = \Omega(x^{0.9})$. Prove your answer.

3. (5+3 points) For the positive integer x, let $n(x)$ denote the number of decimal digits of x. (a) Give a very simple explicit formula for $n(x)$ in terms of the $\log_{10}$ function and the rounding (floor or ceiling) functions. Prove your answer. (b) Prove: $n(x) \sim \log_{10}(x)$ where $\log_{10}$ refers to base-10 logarithms.

4. (3+3 points) Evaluate each of the following sums as a closed-form expression (no summation symbols or dot-dot-dots).

   (a) $\sum_{k=0}^{n} 2^{-k/2}$.

   (b) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} 2^{-k/2}$.

5. (5 points) Find a closed-form expression for the ordinary generating function of the sequence $a_n = 1/(n+1)$ $(n = 0, 1, \dots)$.

6. (3+12+12+6+6+6 points) Let X_n be the number of triangles in a random graph.

   (a) What is the size of the sample space for this experiment?

   (b) Determine $E(X_n)$. Prove your answer. Use indicator variables. A clear definition of these indicator variables accounts for 6 of the 12 points.

   (c) Determine $\mathrm{Var}(X_n)$. Make your expression reasonably simple. (Hint: use the variables introduced in (b).)

   (d) Asymptotically evaluate $\mathrm{Var}(X_n)$. Show that $\mathrm{Var}(X_n) \sim an^b$ where a and b are constants. Determine a and b.

   (e) Asymptotically compare the result with the variance of the sum Y_n of $\binom{n}{3}$ independent indicator variables with the same expectation (1/8). Which of the two is little-oh of the other?

(f) Use the result of (c) to prove the Weak Law of Large Numbers for X_n, i. e., prove that almost surely X_n stays close (within a factor of $(1 \pm \epsilon)$) to its expectation. ("Almost surely" means with probability approaching 1 as $n \to \infty$.)

7. (10 points) Prove that for a strongly connected digraph the following are equivalent:
   (a) the number h divides the length of all cycles;
   (b) the number h divides the length of all closed walks starting at a given vertex x.

8. (4+9B points)
   (a) Define the operator-norm $||A||$ of an $n \times n$ real matrix A.
   (b) (BONUS) Recall the Spectral Theorem: If A is a symmetric real matrix then (A) all eigenvalues of A are real; (B) A has an orthonormal eigenbasis.

   Use the Spectral Theorem to prove that if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the symmetric real matrix A then $||A|| = \max_i |\lambda_i|$.

9. (3+6 points)
   (a) Define stochastic matrices.
   (b) Prove: if z is a (complex) eigenvalue of a stochastic matrix then $|z| \leq 1$.

10. (4+4+4+4+10B points)
    (a) Draw the diagram of a finite Markov Chain which has more than one stationary distribution. Give two stationary distributions for your Markov Chain. Use as few states as possible.
    (b) Recall that a finite Markov Chain is *irreducible* if its transition digraph is strongly connected. Draw the diagram of a non-irreducible finite Markov Chain with a unique stationary distribution. State the stationary distribution. Do not prove. Use as few states as possible.
    (c) Define ergodicity of a finite Markov Chain.
    (d) Draw the diagram of an irreducible but non-ergodic finite Markov Chain.
    (e) (BONUS) Prove: the stationary distribution of an irreducible finite Markov Chain is unique. (Prove uniqueness only. Do not prove the existence of a stationary distribution.) Do not use the Frobenius-Perron Theorem.

11. (10 points) Calculate the determinant of the $n \times n$ matrix $A_n = (a_{i,j})$ where $a_{i,i} = 1$ $(i = 1, \dots, n)$, $a_{i,i+1} = -1$ $(i = 1, \dots, n-1)$, and $a_{i,i-1} = 1$ $(i = 2, \dots, n)$, all other entries are zero. The figure shows the matrix A_5. Hint: Let $d_n = \det(A_n)$. Experiment with small n; observe the pattern, make a conjecture. To prove the conjecture, expand by the last row to obtain a recurrence for the sequence d_n.

$$A_5 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

12. (2+3+5+6+5B points) Consider the Markov Chain defined by the transition matrix

$$B = \begin{pmatrix} 0.4 & 0.6 \\ 0.3 & 0.7 \end{pmatrix}$$

   (a) Draw the transition diagram (the digraph of positive transition probabilities). Label each edge with the corresponding transition probability.

   (b) Is the Markov Chain defined by B ergodic? Why?

   (c) Compute the eigenvalues of B. Show all your work!

   (d) Compute the stationary distribution of this Markov Chain. Give your answer as exact rational numbers in their simplest form. Show all your work!

   (e) (BONUS)(Rapid convergence) Let π denote the stationary distribution and q_n the distribution after n steps. Prove: $||q_n - \pi|| = O(1/10^n)$ (regardless of the initial distribution). Here $||\cdot||$ refers to the standard Euclidean norm.

13. (4 points) Write down the Laplacian of $K_{1,3}$. The vertex of degree 3 should be the first vertex.

14. (8 points) Recall that the "Prüfer code" assigns a sequence $P(T) = (p_1, \ldots, p_{n-2})$ of numbers to every tree on the vertex set $\{1, \ldots, n\}$ and yields a bijective proof of Cayley's formula n^{n-2}. Given T, describe how to construct its Prüfer code. Do not prove its correctness.

15. (5 points) Prove: if G is a planar graph with n vertices then G has an independent set of size $\geq n/6$. (Hint: chromatic number.)

16. (5B points) (BONUS) Prove: if G is a planar graph with n vertices and the complement of G is also planar then G has at most 10 vertices.

17. (5B points) (BONUS) Give a closed-form expression of the **exponential** generating function of the Fibonacci numbers.