CMSC-37110 Discrete Mathematics
FINAL EXAM December 5, 2006

This exam contributes 30% to your course grade.

Do **not** *use book or notes.* **Show all your work.** Prove your answers. If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (2+6+10+3 points) An independent set in a graph $G = (V, E)$ is a subset $S \subseteq V$ such that there are no edges between pairs of vertices in S.
   (a) Count the independent sets in the empty graph $\overline{K}_n$.
   (b) Count the independent sets in the complete bipartite graph $K_{k,\ell}$. Give a simple formula.
   (c) Select an independent set S of $K_{k,\ell}$ at random from the uniform distribution (every independent set has the same probability to be selected). Determine $E(|S|)$. Give a closed-form expression.
   (d) Asymptotically evaluate $E(|S|)$ as $k \to \infty$ and $\ell = k - 5$. Give a very simple expression.

2. (3+7 points)
   (a) Evaluate the sum $\sum_{i=0}^{n} \binom{n}{i} 2^{-i}$. Give a closed-form expression.
   (b) Find the largest term in this sum.

3. (10 points) Prove: if p and q are two distinct odd primes then there exists x such that $x^2 \equiv 1 \pmod{pq}$ but $x \not\equiv \pm 1 \pmod{pq}$.

4. (5 points) What is the probability that a poker hand is a full house? (A poker hand is a random set of five cards out of the standard deck of 52 cards. A full house consists of three cards of a kind and two cards of another kind, for instance three 9s and two queens.) Give a simple expression; do not evaluate.

5. (10 points) For what values of x is the following statement true:
   $(\forall y)($ if $x \mid 12y$ then $x \mid y)$.
   Prove your answer. You must prove (a) that the good values of x are indeed good; and (b) the bad values are indeed bad.

6. (7+3 points)
   (a) Consider the simple random walk on the integers: $X_0 = 0$ and $X_{t+1} = X_t \pm 1$, each possibility having probability 1/2. (The frog flips a coin at each step to decide whether to move right or left by one step.) Compute the probability that $X_{2n} = 0$ (in $2n$ steps the frog is back at the starting point). Give a simple closed-form expression.

(b) Asymptotically evaluate this probability. Give a very simple expression; no factorials or binomial coefficients.

7. (4+3+8B points)

   (a) Let $\{a_n\}$ be a sequence of real numbers. Define, with a well-quantified formula, that $a_n \neq O(1)$. The formula should begin with quantifiers; no Boolean operations (negation, and, or, if-then) should precede any quantifier. No English words permitted.

   (b) Does $a_n \neq O(1)$ mean $|a_n| \to \infty$?

   (c) (Bonus) Let $f(n)$ denote the number of solutions to the equation $x + y = n$ where x and y are prime numbers. For instance, $f(10) = 3$ (the solutions are $3 + 7, 5 + 5, 7 + 3$). Prove that $f(n) \neq O(1)$.

8. (10 points) Let A and B be $k \times n$ matrices over the field F. Prove: $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.

9. (4+2+4+4+4B points) Let T be the transition matrix of a finite Markov Chain.

   (a) Give an example where the limit $\lim_{t\to\infty} T^k$ does not exist. Make your Markov Chain as small as possible. Draw the diagram of the Markov Chain and give the transition matrix.

   (b) Define what it means for a finite Markov Chain to be *ergodic*.

   (c) Assume the limit exists. Show by example that it does not follow that the Markov Chain is ergodic. (Give diagram and transition matrix.)

   (d) Assume the limit exists, $\lim_{t\to\infty} T^k = L$. Prove: every row of L is a stationary distribution.

   (e) (Bonus) Assume the limit exists and the Markov Chain is irreducible (strongly connected). Prove that the Markov Chain is ergodic.

10. (5 points) Among all graphs of diameter 2 with n vertices, which one has the largest independent set? How large is it?

11. (10 points) Let n be an even positive integer and $N = 2^{n/2}$. Prove: almost all graphs on N vertices have no independent set of size greater than n.

12. (10 points) Cayley's formula says that the number of trees on a given set of n vertices is n^{n-2}. Use this result to prove: the number of nonisomorphic trees on n vertices is $> 2.7^n$ (for sufficiently large n).

13. (9+6+8 points) (Variations on the Hat-Checkers problem) n patrons of a theater check their hats. After the show, the hats are distributed at random.

    (a) Determine the probability that nobody gets their own hat. Prove that your result converges to 1/e. Is this convergence fast?

(b) Determine the expected number of "lucky patrons" (those who receive their own hat). Define your random variables!

(c) A secretary is given n envelopes and a list of n addresses. On each envelope he puts a random address; he does not remember which addresses he previously used, so the same person may receive several letters. What is the expected number of those on the list who will not receive the letter? Your answer should be a simple expression. Prove that the answer is asymptotically n/e. Is this convergence fast?

14. (5 points) (Manhattan routes) In the $k \times n$ grid (graph), count the number of shortest paths from the bottom left corner to the top right corner. (Note that the distance between these two points is $k+n-2$.) Give an very simple formula.

15. (8B points)(bonus) Prove: If a prime power p^t divides the binomial coefficient $\binom{n}{k}$ then $p^t \leq n$.

16. (8B points)(bonus) Prove: every tournament on 2^{k-1} vertices contains a subtournament on k vertices which is a DAG. (A tournament is a directed complete graph: every pair of vertices is directed in exactly one direction.)

17. (8B points)(bonus) Prove: for almost all graphs G, $\chi(G) > \omega(G)^{100}$.
($\chi(G)$ denotes the chromatic number and $\omega(G)$ is the clique number, i. e., the size of the largest clique (complete subgraph) in G.)