CMSC-37110 Discrete Mathematics
FINAL EXAM December 4, 2007

This exam contributes 30% to your course grade.

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the proctor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (6+6+6 points) The universe of the variables in each subproblem below is the set of integers, or the subset of integers indicated. For instance, $(\forall x > 0)$ means "for all positive integers x." The gcd notation refers to the *nonnegative* greatest common divisor. Prove your answers.
   (a) TRUE or FALSE? $(\forall x > 0)(\exists y > 0)(\gcd(x, y) = \gcd(x, 2y))$.
   (b) Let $f(x) = \sum_{i=0}^{39} x^i$. Prove: $(\forall x)(f(x) \equiv 0 \text{ or } \pm 1 \pmod{41})$.
   (c) For what values of k is the following statement true: $(\forall x)(\exists y)(\gcd(x, y) = \gcd(x, y + k))$.

2. (10 points) Let $p \neq q$ be odd primes. Prove: $(\exists x)(x^2 \equiv 1 \pmod{pq}$ but $x \not\equiv \pm 1 \pmod{pq})$.

3. (8 points) Suppose you receive twice as many spam email messages as non-spam email messages. Further suppose the probability that a spam email contains the word "free" is $\frac{1}{3}$ and the probability that a non-spam e-mail contains the word "free" is $\frac{1}{30}$. Your software tells you that you received an e-mail that contains the word "free." What is the probability that the email is spam? Carefully formalize the conditions and the probability in question before you attempt to compute it. Express your answer as a fraction reduced to lowest terms.

4. (4+14+2 points) Let R and S be two spanning trees of the complete graph on the vertex set $[n]$, chosen independently and uniformly from among all spanning trees. (a) What is the size of the sample space for this experiment? (b) Determine the expected number of common edges of X and Y. Your answer should be a very simple closed-form expression. Carefully define your random variables. (c) Asymptotically evaluate your answer to question (b) (obtain an even simpler expression asymptotically equal to your answer).

5. (6 points) If a strongly connected digraph G has period $h \geq 2$, prove that the value of h determines the chromatic number of G. Give a clear statement of what the chromatic number is for each value of h. (Legal colorings of a digraph ignore orientation: if $v \to w$ then the vertices v and w must have different color.)

6. (4 points) Let $\pi(x)$ denote the number of primes $\leq x$. Recall that the Prime Number Theorem (PNT) says $\pi(x) \sim x/\ln x$. True or false: $\pi(x) = O(x^{0.9})$? Prove your answer.

7. (6 points) We wish to distribute n identical chocolate coins among k children so that each child receives at least 2 chocolate coins. Count the possible outcomes. (An "outcome" is defined by the number of chocolate coins received by each child.) Prove your answer.

8. (4+4 points) (a) Suppose the sequence $\{a_n\}$ satisfies the relation $a_{n+1} = O(a_n)$. Prove: the generating function of the sequence has positive convergence radius, i. e., $(\exists r > 0)$ such that the series $\sum_{n=0}^{\infty} a_n x^n$ converges for all x with $|x| < r$. (b) Prove that the converse is false, i. e., a positive convergence radius does not guarantee $a_{n+1} = O(a_n)$.

9. (6 points) Which bipartite graphs have diameter 2? (Describe all.) Your answer should be very simple. Prove it.

10. (10 points) Calculate the determinant of the $n \times n$ matrix $A_n = (a_{i,j})$ where $a_{i,i} = 1$ $(i = 1, \ldots, n)$, $a_{i,i+1} = -1$ $(i = 1, \ldots, n-1)$, and $a_{i,i-1} = 1$ $(i = 2, \ldots, n)$, all other entries are zero. The figure shows the matrix A_5. Hint: Let $d_n = \det(A_n)$. Experiment with small n; observe the pattern, make a conjecture. To prove the conjecture, expand by the last row to obtain a recurrence for the sequence d_n.

$$A_5 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

11. (8 points) Prove: if G is a planar graph with n vertices then G has an independent set of size $\geq n/6$. (Hint: chromatic number. Prove the required lemma about the chromatic number of planar graphs. Use the fact that every planar graph has a vertex of degree ≤ 5; do not prove this fact.)

12. (9+6B points) Let G be a connected undirected graph with n vertices and diameter d. Let A be the adjacency matrix of G. (a) Prove that the matrices $I, A, A^2, \ldots, A^d$ are linearly independent over $\mathbb{R}$ (as vectors in $\mathbb{R}^{n^2}$). (b) (Bonus) Prove that under the same conditions, A has at least $d+1$ distinct eigenvalues.

13. (7+5B points) (a) Prove: all eigenvalues of a real symmetric matrix are real. (Do not use the Spectral Theorem.) (b) (Bonus) Recall that a real $n \times n$ matrix A is *orthogonal* if $A^T = A^{-1}$. Prove: all complex eigenvalues of a real orthogonal matrix have unit absolute value.

14. (4+4+10 points) (a) Define the operator-norm $||A||$ of an $n \times n$ real matrix A. (b) State the Spectral Theorem. (c) Use the Spectral Theorem to prove that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the symmetric real matrix A then $||A|| = \max_i |\lambda_i|$.

15. (3+9+6B points) (a) Define stochastic matrices. (b) Prove: if λ_i is a (complex) eigenvalue of a stochastic matrix then $|\lambda_i| \leq 1$. (c) (Bonus) Prove: if $\lambda_1 = 1$ and $(\forall i \geq 2)(|\lambda_i| < 1)$ then the corresponding Markov Chain is ergodic (irreducible and aperiodic).

16. (Bonus problem, 6B points) Prove that for almost all graphs, every vertex has degree $n/2 + O(\sqrt{n \log n})$.