CMSC-37110 Discrete Mathematics
FIRST MIDTERM EXAM October 23, 2007

This exam contributes 16% to your course grade.

Do **not** *use book, notes.* You **may** use a calculator for *basic arithmetic* but not for more advanced functions such as g.c.d. **Show all your work.** If you are not sure of the meaning of a problem, **ask the proctor.**

1. (4+4 points) Compare the following two statements about a sequence $\{a_n\}$ of real numbers:

   (a) $\lim_{n\to\infty} a_n = \infty$;

   (b) $(\exists n_0)(\forall n \geq n_0)(a_{n+1} > a_n)$.

   Prove that (b) is neither necessary nor sufficient for (a). Clearly divide your answer into two parts and indicate which part disproves necessity and which disproves sufficiency. In each part, clearly state the goal.

2. (3+3) True or false. Prove your answers.

   (a) $2^{\binom{n}{2}} \sim 2^{n^2/2}$

   (b) $\ln(n \ln n) \sim \ln n$

3. (3+3+5 points) Give closed-form expressions for the following sums:

   (a) $\sum_{k=0}^{n} 2^{k/2}$.

   (b) $\sum_{k=0}^{n} \binom{n}{k} 2^{k/2}$.

   (c) $\sum_{k=0}^{n} k\binom{n}{k}$. (Prove your answer.)

4. (6 points) Decide whether or not the following system of congruences is solvable. If solvable, do not solve, just prove your answer.

   $x \equiv 4 \pmod 8$

   $x \equiv 6 \pmod{10}$

   $x \equiv 1 \pmod 5$

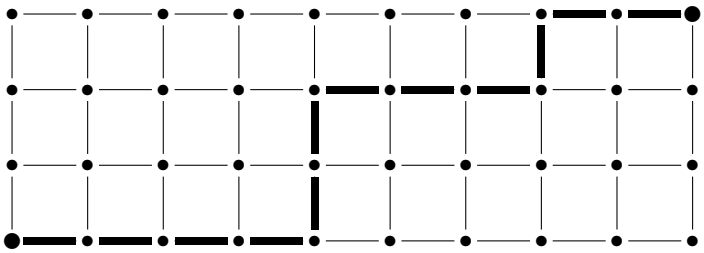
5. (4 points) Find the remainder of $1! + 2! + \ldots + 99! + 100!$ when divided by 18.

6. (7 points) True or fale? Prove your answer. Do NOT use a calculator. Show all your work.

   "If g.c.d.$(x, 35) = 1$ then $x^{34} \equiv 1 \pmod{35}$."

7. (6 points) Let F_n denote the n-th Fibonacci number. (Recall that $F_0 = 0$ and $F_1 = 1$.) Prove: $(\forall n \geq 0)(F_n < 1.7^n)$. (Help: $17^2 = 289$.) Do not use calculator, and do not use the explicit formula for Fibonacci numbers. Use mathematical induction.

8. (6 points) Count the shortest paths from the bottom left corner to the top right corner of the $k \times \ell$ grid graph. The figure shows the 4×10 grid graph with a shortest path between the given corners highlighted. Your answer should be a very simple closed-form expression (no summation or dot-dot-dots). Prove your answer.



9. (2+4 points) Flip a coin n times. (a) What is the probability of getting exactly 2 heads? (b) What is the probability of getting at least 2 heads? Your answers should be simple closed-form expressions (no summation symbol, no "dot-dot-dots").

10. (5+2 points) Roll two 6-sided dice. (a) What is the probability that the first roll is 4 given that the sum of the dice is 8? (b) Let A be the event that the first roll is 4. Let B be the event that the sum of the dice is 8. Are events A and B independent, positively correlated, or negatively correlated?

11. (8 points) Let X_n be the number of of occurrences of the string "TEST" in a random string of length n over the English alphabet of 26 (upper case) letters. ("TEST" needs to appear as 4 consecutive letters, like in DCCTESTGHA.) For what value of n is $E(X_n) = 1$? Prove your answer. Give a clear definition of the random variables involved. Clarity of this definition accounts for half the grade.

12. (Bonus problem, not required, 6 points) Construct n events that are not independent but every $n-1$ of them are independent. Use the smallest possible sample space. For 2 extra bonus points, prove that you sample space is smallest possible.

13. (Challenge problem, no point value) Prove:
$(\forall k \geq 1)(\exists x)(x^2 \equiv -1 \pmod{5^k})$.

14. (Challenge problem, no point value) We have a deck of n cards, numbered $\{1, 2, \ldots, n\}$. A "hand" consists of k cards. What is the probability that a hand has no consecutive numbers? Your answer should be a very simple closed-form expression (a quotient of two binomial coefficients).