CMSC-37110 Discrete Mathematics
SECOND MIDTERM EXAM November 13, 2007

This exam contributes 16% to your course grade.

Do **not** *use book, notes.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the proctor.**

1. (8 points) Let G be a connected planar graph such that every vertex has degree ≥ 3. John claims to have drawn G in the plane such that each region is 6-sided. Prove him wrong.

2. (6 points) Prove that every DAG can be topologically sorted. In other words: if G is an acyclic digraph, prove that the vertices of G can be numbered such that if $i \to j$ is an egde then $i < j$.

3. (9 points) Prove that almost all tournaments are strongly connected. In other words, let p_n denote the probability that a random tournament[1] on a given set of n vertices is strongly connected. Prove that $p_n \to 1$. State the size of the sample space for this experiment.

4. (7+3 points) Let A be the adjacency matrix of a DAG G with n vertices. (a) Determine the rank of A^n. Prove your answer. (b) Show that the rank of A^{n-1} determines whether or not G has a Hamilton path.

5. (9+3 points) Let G be a random graph on the vertex set $[n]$ (each pair of vertices is adjacent with probability $1/2$). Let p_n denote the probability of the event that $\deg(1) = \deg(2)$. (a) Give a closed-form expression for p_n. (b) Prove that $p_n \sim c/\sqrt{n}$. Determine the value of the constant c.

6. (1+5+2+8+3 points) Let T_n denote the number of triangles in a random graph. (a) State the size of the sample space in this experiment. (b) Write T_n as a sum of indicator variables. Give a clear definition of each indicator variable in the sum and state the number of indicator variables used. (c) Determine $E(T_n)$. (d) Determine $\mathrm{Var}(T_n)$. Give a closed-form expression. (e) Asymptotically evaluate $\mathrm{Var}(T_n)$: show that $\mathrm{Var}(T_n) \sim cn^d$; determine the constants c and d.

7. (Bonus problem, not required, 6 points) Let G be a directed graph such that all vertices have out-degree $\leq k$. Prove that G is $(2k+1)$-colorable. (A legal coloring of a directed graph is defined the same way as for undirected graphs; coloring is insensitive to the direction of edges.)

8. (Bonus problem, not required, 6 points) Prove that for almost all graphs, every vertex has degree $n/2 + O(\sqrt{n \log n})$.

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[1] A *tournament* is an orientation of the complete graph, i. e., every edge of the complete graph is oriented in one of the two possible directions. In a *random tournament,* these orientations are assigned independently with each choice having probability $1/2$.