CMSC-37110 Discrete Mathematics
QUIZ November 27, 2007

Name (print): ____________________________________

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. **Write your solution in the space provided.** You may continue on the reverse. This exam contributes 8% to your course grade.

1. (3+5+2+4+6B points) (a) Given a Finite Markov Chain (MC) by its $n \times n$ transition matrix T, define what it means for a probability distribution $\mathbf{q} = (q_1, \dots, q_n)$ to be *stationary.* (b) Draw an example of a weakly connected MC with more than one stationary distribution. Use the smallest possible number of states. State two stationary distributions of your MC. (c) Indicate why the number of states you used is minimal. (d) Prove: if a MC has more than one stationary distribution then it has infinitely many. (e) BONUS PROBLEM. Prove: if T is *irreducible* (the transition digraph is strongly connected) then there cannot be more than one stationary distribution.

2. (11+4B points) (a) Let A be the adjacency matrix of the complete graph K_n. Compute $\det(A)$. Show and explain each step you make. (b) BONUS PROBLEM. Recall that a *derangement* is a permutation which does not fix any element of its domain. Among the derangements of n elements, are the even or the odd permutations the majority? (Your answer may depend on n. Prove your answer.)

3. (10+5+5B points) Let A be the adjacency matrix of a DAG. (a) Prove that zero is the only eigenvalue of A. (b) Find the characteristic polynomial of A. Prove your answer. (c) BONUS PROBLEM: For every n, construct a DAG with n vertices such that its adjacency matrix does not have two linearly indepedent eigenvectors. Prove your example is correct.

4. (7B points) BONUS PROBLEM. Let $N = 2^n$ and let $A_1, \dots, A_N$ be the subsets of $[n]$. Consider the $N \times N$ matrix $B = (b_{ij})$ where $b_{ij} = |A_i \cap A_j|$. Prove: $\mathrm{rk}(B) = n$. (5 points for $\mathrm{rk}(B) \leq n$ and 2 points for $\mathrm{rk}(B) \geq n$. Each part is an "AH-HA problem" with a 2-line solution.)