Advanced Combinatorics
Math 295, Spring 2007
Class Summary and Homework Questions

Lecture 1 - March 26

All homework listed below is due Wednesday, March 28, before class. Problems labeled "Do" should not be handed in; do them for your own benefit to check your understanding of the concepts. "Puzzle problems" have no deadline except they expire when discussed in class, so if you are working on some of those problems, let me know so as to give you time. You also may ask for a hint.

READING: Chapters 1 and 2 of the instructor's "Discrete Mathematics" lecture notes (DM) (Quantifier notation, Asymptotic notation; skip section 2.6 (partitions)). Study sections 2.1 to 2.6 in the Matoušek-Nešetřil text (MN).

Asymptotic equality of sequences, the Prime Number Theorem, Stirling's Formula, Chebyshev's weaker version of the Prime Number Theorem, little-oh notation.
Do: Show $a_n \sim b_n$ if and only if $a_n - b_n = o(a_n)$.
The logarithmic integral.
Do: Prove: $(\forall \varepsilon > 0)(\ln x = o(x^\varepsilon))$.
Riemann Hypothesis: $|\pi(x) - \mathrm{li}(x)| = O(\sqrt{x})$.
Do: Show $\mathrm{li}(x) \sim \frac{x}{\ln x}$.
Polynomial growth versus exponential growth.
Do: For a polynomial $p(x) = a_n x^n + \cdots + a_0$, where $a_n \neq 0$, show that $p(x) \sim a_n x^n$.
Big-Oh notation.
Do: Show that if $a_n = o(b_n)$ then $a_n = O(b_n)$.
Big-Omega and big-Theta notation.
Do: If $a_n \sim b_n$ then $a_n = \Theta(b_n)$.
Homework 1.1: Suppose $a_n, b_n > 1$. Consider the statements (1) $a_n \sim b_n$ and (2) $\ln a_n \sim \ln b_n$. Show (a) that (1) does not imply (2), and (b) if, in fact, $a_n > 1 + \varepsilon$ for some fixed positive ε then (1) does imply (2).

Application of above homework to Stirling's Formula: $\ln(n!) \sim n \ln n$.
Divisibility, greatest common divisor.
Do: Show that $\lim_{x\to\infty} \lfloor x/4 \rfloor / x = 1/4$
$d(n)$ is the number of positive divisors of a number n.
Do: Show that for $n = \Pi p_i^{k_i}$,

$$d(n) = \prod_{i=1}^{r} (k_i + 1).$$

Hint: Prove if a and b are relatively prime, then $d(ab) = d(a)d(b)$.
Binomial coefficients, binomial theorem.
Do: Prove that $\binom{2n}{n}/4^n \sim c/\sqrt{n}$ for some constant c. Determine the value of c.

Do: Prove if p is prime, $p^r \mid n!$, and $p^{r+1} \nmid n!$, then

$$r = \sum_{i=1}^{\infty} \lfloor n/p^i \rfloor.$$

Homework 1.2: Prove that if p is prime and

$$p^t \mid \binom{n}{k}$$

then $p^t \le n$.
Puzzle: Find a appropriate bijection to prove that the number of even subsets of a nonempty set is equal to the number of odd subsets.
Homework 1.3: MN 2.1.4 (p52).
Homework 1.4: MN 2.2.1 (p54).
Puzzle: Give two proofs (a bijective proof and an algebra proof) of the following identity:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Do: Prove that $n! > (n/\mathrm{e})^n$ using the power series expansion $\mathrm{e}^x = \sum_{n=0}^{\infty} x^n/n!$.
Do: If $1 \le k \le n$ then

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{\mathrm{e}n}{k}\right)^k.$$

Hint: for the more difficult part (right-hand inequality) use the preceding problem.

Lecture 2 - March 27

The Homework below is due on Thursday, March 29th.

Pascal's Identity and Pascal's Triangle, three proofs of (# odd subsets) = (# even subsets), proof of the Binomial Theorem. Major result: Chebyschev's "weak Prime Number Theorem" which says that $\pi(x) = \Theta(x/\ln x)$. Proof of the upper bound part, i. e., $\pi(x) = O(x/\ln x)$.
Main Lemma: $\prod_{p \le x} p < 4^x$. (p denotes primes.)
Sublemma. $\prod_{k+1 \le p \le 2k+1} p \mid \binom{2k+1}{k}$.

Homework 2.1: Prove the inequality

$$\binom{2k+1}{k} < 4^k.$$

Do: Prove if $0 < a_n \lesssim b_n$, then $a_n = O(b_n)$. (Check the definition and basic properties of the "$\lesssim$" relation in DM, Chapter 2 (Asymptotic notation).)

Ramsey Game, Erdős-Rado arrow notation, Ramsey's Theorem, Erdős-Szekeres bound: $\binom{k+\ell}{k} \to (k+1, \ell+1)$.
Homework 2.2: Show $k^2 \nrightarrow (k+1, k+1)$.
Homework 2.3: Extend the Erdős-Rado arrow notation to give meaning to

$n \to (k, \ell, m)$ and prove $17 \to (3,3,3)$.
Puzzle*: Show $16 \nrightarrow (3,3,3)$. Hint: Consider the field of order 16.
Puzzle: prove: the product of k consecutive integers is always divisible by $k!$.

Lecture 3 - March 28

The Homework below is due on Monday, April 2.

Chebyshev from below: $\pi(x) = \Omega(x/\ln x)$; the proof uses the fact that $p^t \mid \binom{n}{k} \Rightarrow p^t < n)$. Bertrand's Postulate: $(\forall n > 1)(\exists \text{ prime } p)(n < p < 2n)$. (Conjectured by Bertrand (1845), proved by Chebyshev (1850).)
Puzzle: Denote by p_n the nth prime. Show that the Prime Number Theorem is equivalent to the statement $p_n \sim n \ln n$.
Do: Prove that the Prime Number Theorem implies Bertrand's Postulate for all sufficiently large n.
Puzzle: Assuming $\pi(x) < cx/\ln x$ for some $c < 2\ln 2$, show that Bertrand's Postulate follows for all sufficiently large n.

READING (by Monday): DM Chapter 7 (Finite Probability Spaces) Sections 7.1 and 7.2, including independence of events and conditional probability.
Do: DM 7.1.9, 7.1.11, 7.1.14.
Puzzle: DM 7.1.19.
Homework 3.1: DM 7.1.17.
Definitions of a finite probability space (Ω, P), uniform probability distributions, an elementary event, an event, a random variable.
Expected value of the random variable X: $E(X) = \sum_{x\in\Omega} X(x)P(x)$.
$E(X) = \sum_{y\in\mathbb{R}} yP(X = y)$.
Linearity of expectation: $E(cX) = cE(X)$, $E(X + Y) = E(X) + E(Y)$.
Indicator variables, expected value of the number of heads in n coin flips (2 proofs).
Homework 3.2: DM 7.2.13
Searching Homework 3.3: Find in MN and understand the proof of

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k-1} + \binom{n}{k} < \left(\frac{en}{k}\right)^k.$$

(Hand in the page number.)
Puzzle: For two randomly chosen positive integers a and b, prove that

$$P(a \text{ and } b \text{ are relatively prime}) = \frac{6}{\pi^2}$$

under the assumption that the probability exists (i.e., the limit that defines this probability exists). Should take 2 lines, using the following result.
Searching Homework 3.4: Find in MN and understand the proof of the following result, due to **Euler:**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(Hand in the page number.)

Lecture 4 - March 29

The Homework below is due on Monday, March 2 except where indicated otherwise.

The grading scheme for the class will be as follows:

Quiz 1	15%
Quiz 2	15%
Quiz 3	20%
Homework	40%
Class Participation	10%

Unimodal sequence: a sequence $a_0, \dots, a_n$ such that $a_0 \le a_1 \le \dots \le a_k \ge \dots \ge a_{n-1} \ge a_n$ (the sequence increases and then decreases). Note that all *monotone* sequences are unimodal (an increasing sequence will have $k = n$, a decreasing sequence will have $k = 0$).

Bernoulli trials (biased coin flips) and the binomial distribution:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Homework 4.1: (a) Prove the numbers $P(X = k)$ above (for $k = 0, 1, \dots, n$) define a unimodal sequence.

(b) Which term is the largest (as a function of n and p)? (Prove your answer.)

Homework 4.2: Find a random variable X such that $X > 0$, $E(X) = 10^6$ and $P(X = 1) = 0.99$.

Puzzle: If $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ is a polynomial such that all the roots of f are negative real numbers, show the sequence $1, a_{n-1}, a_{n-2}, \dots, a_0$ is unimodal.

Do: DM 5.1.14, 5.1.20.

Theorem (Erdős) $2^{\frac{k}{2}} \nrightarrow (k+1, k+1)$

Proof: non-constructive, inaugurated the probabilistic method in combinatorics (1949).

Do: For events $D_1, \dots, D_n$, prove the "union bound": $P(\bigcup_{i=1}^n D_i) \le \sum_{i=1}^n P(D_i)$.

Solved exercise: There are more ten digit numbers containing exactly one 9 than those with no 9's at all. (2 solutions.)

Homework 4.3: Find a number $C > 1$ such that $C^k \nrightarrow (k+1, k+1, k+1)$. Make your C as large as you can. (Hint: Mimic Erdős's proof.)

Homework 4.4: (due Tuesday, March 3) Let $A_1, \dots, A_m$ be n-subsets of a set X. Prove: if $m \le 2^{n-1}$ then it is possible to color the elements of X red/blue so that none of the A_i becomes monochromatic (each A_i will have elements of both colors). (Hint: probabilistic method.) **Originally, the bound was erroneously stated as 2^n.** My apologies. The deadline is extended to Wednesday, March 4.

Homework 4.5: (due Tuesday, March 3) Suppose $k_n = o(\sqrt{n})$.

Prove that $\binom{n}{k_n} \sim \frac{n^{k_n}}{k_n!}$.

Homework 4.6: (due Tuesday, March 3) Let us pick a number x uniformly from $\{1, \dots, n\}$. For exactly what values of n are the following events independent: event A : "x is even;" and event B : "x is divisble by 3."

Puzzle: Let $S(n,k)$ be the number of those subsets of an n-set which have size divisible by k. In other words, $S(n,k)$ is the sum of those binomial coefficients $\binom{n}{t}$ where $k \mid t$. (a) Prove: $|S(n,3) - 2^n/3| < 1$. (b) For what values of n is $S(n,4) = 2^{n-2}$?