**Advanced Combinatorics**
**Math 295, Spring 2007**
Class Summary and Homework Questions: Week 2

**Lecture 5 - April 2**

The Homework below is due on Wednesday 4th March.

Explicit bound on the relative error in Stirling's Formula, valid for all $n \geq 1$: $n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\, e^{\varepsilon_n}$ where

$$\frac{1}{12n+1} < \varepsilon_n < \frac{1}{12n}.$$

Note that for small $|x|$, the value $e^x$ is close to but greater than $1 + x$. This means that the relative error in Stirling's formula is close to but possibly slightly greater than $1/12n$. (Note the error in the version stated in class.)
Explicit bound on the largest binomial coefficient (true for all $n, k$, where $0 \leq k \leq n$ and $n \geq 1$:

$$\binom{n}{k} < \frac{2^n}{\sqrt{n}}. \tag{1}$$

**Do:** Find this result for even $n$ in MN. Prove it for all $n \geq 1$.
Note that for even $n$, $\binom{n}{n/2} \sim \sqrt{2/\pi}\, 2^n/\sqrt{n}$. This is an asymptotic result. $\sqrt{2/\pi} = 0.79788... < 1$, so asymptotically a stronger inequality holds than (1).
**Homework 5.1:** Let $X$ be the number of heads obtained in $n$ flips of a coin and let $Y$ be the number of heads in an additional $n$ flips of the coin.
(a) Find a closed-form expression for $P(X = Y)$ (i. e., with no $\sum$ or $\ldots$).
(b) Prove $P(X = Y) \sim an^b$ and determine $a$ and $b$.
(c) Prove $P(|X - Y| \leq k) < \frac{2k+1}{\sqrt{n}}$.
Two events $A$ and $B$ are independent if $P(A \cap B) = P(A)P(B)$. Positive and negative correlation, relation of independence of events to conditional probability, independence of more than two events: $A_1, \ldots, A_n$ are independent if, for all $I \subseteq \{1, 2, \ldots, n\}$,

$$P(\bigcap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$$

**Do:** Show that $A$ is independent of itself if and only if $A$ is trivial. (An event is trivial if it is either $\emptyset$ or $\Omega$.)
**Do:** Suppose $A$, $B$ and $C$ are independent. Show that $A$ and $B \cup C$ are independent and $A$, $B$ and $\overline{C}$ are independent.
**Do:** Show that $A$, $B$ and $C$ being independent does not guarantee that $A \cup B$ and $B \cup C$ are independent.
**Do:** Suppose $A$, $B$, $C$ and $D$ are independent. Show
(a) $A$, $B \cup C$ and $D \cap E$ are independent, and
(b) $(A \cup \overline{B}) \cap C$ and $D \cup \overline{E}$ are independent.
**Do:** Formulate the general principle of which the above exercises are special cases. ("Boolean combinations of disjoint sets of independent events are independent." Make this more precise.)
Independence of random variables.
**Do:** Show that the indicator random variables $\theta_{A_1}, \ldots, \theta_{A_n}$ are independent if

and only if the events $A_1, \dots, A_n$ are independent.
**Theorem (multiplicativity of expectation)** If $X_1, \dots, X_n$ are independent random variables then

$$E(\prod_{i=1}^{n} X_i) = \prod_{i=1}^{n} E(X_i). \tag{2}$$

**Do:** Show that this is not true if we relax the requirement of independence to *pairwise independence*. Give a counterexample with three random variables and the smallest possible sample space.
Covariance of random variables $X$ and $Y$: $\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$.
We say that $X$ and $Y$ are *positively correlated* if $\text{Cov}(X, Y) > 0$; *negatively correlated* if $\text{Cov}(X, Y) < 0$, and *uncorrelated* if $\text{Cov}(X, Y) = 0$.
**Homework 5.2:** Construct two random variables which are uncorrelated but not independent. Find the minimum size sample space $|\Omega|$ needed to exhibit this.
Variance: $\text{Var}(X) = E((X - E(X))^2) = E(X^2) - (E(X))^2 = \text{Cov}(X, X)$.
$\text{Var}(X) \geq 0$; it is equal to 0 if and only if $X$ is constant.
Standard deviation: $\sigma(X) = \sqrt{\text{Var}(X)}$.
Cauchy-Schwarz inequality: $E(X^2) \geq (E(X))^2$.
Variance of a sum of random variables:
If $X = \sum_{i=1}^{n} Y_i$ then

$$\text{Var}(X) = \sum_{i=1}^{n} \text{Var}(Y_i) + \sum_{1 \leq i < j \leq n} \text{Cov}(Y_i, Y_j). \tag{3}$$

(Note that the number of terms in the second sum is $\binom{n}{2}$.)
**Corollary (additivity of variance)** If $Y_1, \dots, Y_n$ are pairwise random variables then $\text{Var}(\sum_{i=1}^{n} Y_i) = \sum_{i=1}^{n} \text{Var}(Y_i)$.
Corollary. If $X$ is the number of 1s in a sequence of $n$ Bernoulli trials with probability $p$ then $\text{Var}(X) = p(1-p)n$. Therefore the standard deviation of the number of heads in a sequence of $n$ coin flips is $\sqrt{n}/2$ ("law of errors").
**Reading:** From DM: Independence of random variables, variance, Markov's Inequality and Chebyshev's Inequality.
**Do:** Prove: If there exist $n$ non-constant independent random variables over the probability space $(\Omega, P)$ then $|\Omega| \geq 2^n$.
**Puzzle:** Construct $n$ pairwise independent $(0,1)$-variables with expectation $1/2$ over a sample space of size $(n + 1)$ assuming $n = 2^k - 1$.
Oddtown and Eventown.
**Do:** Find this subject in one of your texts.
**Homework 5.3:** The inhabitants of Oddtown (population $n = 100$) like to form clubs. They have rules to govern their formation. The clubs in Oddtown must satisfy:
(1) Each club has an odd number of members;
(2) Each pair of clubs shares an even number of members.
Find a *maximal set* of Oddtown clubs consisting of only two clubs. "Maximal" means no further club can be added without violating the Oddtown rules. (Contrast this with the *maximum* number of clubs possible in Oddtown; that number is $n$.)
**Puzzle.** The maximum number of clubs in Eventown is $2^{\lfloor n/2 \rfloor}$.
**Puzzle.** Every maximal set of Eventown clubs is maximum.

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**Lecture 6 - April 3**

The Homework below is due on Thursday, March 5.

$\mathbb{R}^n$, vectors, scalars, linear combinations, linear independence, span, rank of a set of vectors, row-rank and column-rank of a matrix.
**Miracle #2 of Linear Algebra:** row-rank = column-rank.
**Miracle # 1 of Linear Algebra:** (impossibility of boosting linear independence): If $a_1, \ldots, a_r$ are linearly independent and $a_i \in \text{span}(b_1, \ldots, b_s)$ for all $i = 1, \ldots, r$ then $r \leq s$.
Subspaces: $U \subseteq \mathbb{R}^n$ is a subspace (denoted $U \leq \mathbb{R}^n$) if $U$ is closed under linear combinations.
**Do:** For any subset $T \subseteq \mathbb{R}^n$, $\text{span}(T) \leq \mathbb{R}^n$ (i.e. $\text{span}(T)$ is a subspace).
**Do:** Show that $U \leq \mathbb{R}^n$ if and only if $\text{span}(U) = U$.
Dimension of a subspace.
**Do:** For $a_1, \ldots, a_r \in \mathbb{R}^n$, define an *elementary transformation* as follows. Fix $i$ and $j$, $i \neq j$ and $\alpha \in \mathbb{R}$, and replace $a_i$ with $a_i' = a_i - \alpha a_j$ and for $k \neq i$ set $a_k' = a_k$. Prove $\text{rk}(a_1, \ldots, a_n) = \text{rk}(a_1' \ldots, a_n')$.
**Do:** Prove that elementary row-transformations do not change either the row-rank or the column rank of a matrix; same about elementary column transformations.
**Do:** Prove that a permutation of the rows or columns does not change the rank.
**Do: (Gaussian elimination)** Prove a sequence of elementary row- and column-transformations and column permutations reduces any matrix to a matrix with the first $r$ diagonal entries being non-zero and all other entries being zero. (Thus, the rank of such a matrix is $r$.)
**Do:** Use Gaussian elimination to prove Miracle # 2.
Entire theory above works with $\mathbb{R}$ replaced with a field, for example $\mathbb{C}$, $\mathbb{Q}$ or $\mathbb{F}_p$ ($p$ prime), Vector space over a field $F$, $\dim_F(F^n) = n$.
Basis for a subset $T$ is a maximal linearly independent subset of $T$.
**Do:** All bases for $T$ have the same size ($= \text{rk}(T)$).
**Do:** Prove $B$ is a basis for $T$ if and only if $B$ is linearly independent and $\text{span}(B) = \text{span}(T)$.
The dot product in $F^n$: $a \cdot b = \sum_{i=1}^n a_i b_i$.
**Oddtown Theorem:** The number of clubs in Oddtown is no more than the population.
Lemma: The incidence vectors of Oddtown clubs are linearly independent over $\mathbb{F}_2$.
Observation: the dot product of the incidence vectors $c_A$ and $c_B$ ($A, B \subseteq \{1, \ldots, n\}$) is $|A \cap B|$. Lemma: If $\mathbf{d}_1, \mathbf{d}, \mathbf{d}_m \in F^n$ are such that $\mathbf{d}_i \cdot \mathbf{d}_j = 0$ if $i \neq j$ and $\mathbf{d}_i \cdot \mathbf{d}_i = 1$, then $\mathbf{d}_1, \ldots, \mathbf{d}_n$ are linearly independent.
**Puzzle:** Find* all the values of $p$ such that there exists a non-trivial self perpendicular vector in $\mathbb{F}_p^2$. *Do so empirically to discover the very simple pattern, then prove your guess — this bit is hard.
Subfields.
**Do:** If $F$ is a subfield of $G$ and $A$ is a matrox over $F$ then $\text{rk}_F(A) = \text{rk}_G(A)$. (Hint: Gaussian elimination does the exact same thing over both fields.) **Homework 6.1:** Find a $3 \times 3$ $(0,1)$-matrix $A$ such that $\text{rk}_{\mathbb{R}}(A) \neq \text{rk}_{\mathbb{F}_2}(A)$.
**Puzzle:** If $A$ is an integral matrix (the entries are integers), show $\text{rk}_{\mathbb{F}_p}(A) \leq$

$\text{rk}_{\mathbb{R}}(A)$. Hint: Use the last "do" below.

**Puzzle (Generalized Fisher inequality):** Fix $k \geq 1$. Let us replace the rules of Oddtown with the rules $|C_i \cap C_j| = k$ and $C_i \neq C_j$ for all $i \neq j$. Prove that we still have $m \leq n$. Hint: Prove that the incidence vectors of the clubs are linearly independent over $\mathbb{R}$.

**Homework 6.2:** For $k \times \ell$ matrices $A$ and $B$ over a field $F$, prove that $\text{rk}(A + B) \leq \text{rk}(A) + \text{rk}(B)$.

**Puzzle:** Given 13 weights (real numbers) suppose if we remove any one we can partition the 12 remaining into two sets of six of equal total weights. Prove all the weights are equal. Hint: First assume the weights are integers.

The *determinant* of a matrix: $\det\colon F^{n\times n} \to F$ is a function given by

$$\det(A) = \sum_{\sigma \in \text{perm}} \text{sign}(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)}$$

where $\text{sign}(\sigma) = \pm 1$ depending on the parity of the number of transpositions (pairwise switches) that combine to $\sigma$.

**Puzzle.** Prove that $\text{sign}(\sigma)$ is well defined (depends only on $\sigma$ and not on the particular sequence of transpositions chosen).

**Do:** An elementary transformation does not change the value of the determinant.

Definition: An $n \times n$ matrix is **non-singular** if its columns are linearly independent, i. e., if its rank is $n$.

**Do:** Prove: the $n \times n$ matrix is non-singular if and only if its determinant is not zero.

Def: **Determinant rank:** The determinant-rank of a matrix $A$ is the largest $r$ such that $A$ contains an $r \times r$ non-singular submatix.

**Do:** Prove that the rank of $A$ is the same as its determinant-rank. (Hint: prove that elementary transformations don't change the determinant-rank.)