Advanced Combinatorics
Math 295, Spring 2007
Class Summary and Homework Questions: Week 2, part 2

Lecture 7 - April 4th

The Homework below is due on Tuesday, March 10.

Do: (a) Prove that $a_n > 0$ and $a_n \to 0$ as $n \to \infty$ does not imply $n^{a_n} \to 1$.
(b) If $c_n \to \infty$, prove there exists a sequence $(b_n)_n$ such that $b_n > 1$, $b_n \to 1$ and $b_n^{c_n} \to \infty$.
A *permutation* f is a bijection $f\colon A \to A$ to itself, S_n is the set of all permutations of $\{1, 2, \ldots, n\}$, cyclic notation for permutations and k-cycles.
Puzzle: Prove, for two permutations f and g, fg has the same cycle structure as gf.
Inverse of a permutation g^{-1} is a permutation such that gg^{-1} is the identity permutation, an involution is a permutation g for which $g^2 = \text{id}$, the support of a permutation: $\text{supp}(f) := \{x \mid x^f \neq x\}$, two permutations are disjoint if their supports are disjoint.
Do: If f and g are disjoint show they commute: $fg = gf$.
A permutation $f \neq \text{id}$ commutes with itself and $f^l f^k = f^k f^l$ despite having non-zero intersection of supports.
Puzzle: Find permutations f and g which are not disjoint and commute but do not arise via the examples above.
A transposition is a 2-cycle, a set $T \subset S_n$ generates $g \in S_n$ if g can be written as $g = a_1 a_2 \ldots a_n$ where $a_i \in T$ for all $i = 1, \ldots, n$ (perhaps with repetition).
Do: Prove every permutation can be written as a product of disjoint cycles uniquely up to order.
Theorem. Transpositions generate all permutations.
A k-cycle is a product of a $k-1$ transpositions.
Do: The transpositions $(12), (23), \ldots, (n-1, n)$ generate S_n.
A permutation is *even (odd)* if it is the product of an even (odd) number of transpositions.

$$\text{sgn}(f) = \begin{cases} 1, & \text{if } f \text{ is even} \\ -1, & \text{if } f \text{ is odd} \end{cases}$$

This definition completes the definition of the determinant.
Do: Prove: exactly half of S_n is even (for $n \geq 2$).
Puzzle: Prove that if a permutation is even it is not odd. Hint: This is equivalent to proving the identity is not odd.
Sam Lloyd's Puzzle. Given a 4×4 square and 15 unit squares we may slide the 15 unit squares around inside the large square.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	×

A permutation of $\{1, \ldots, 15\}$ (leaving × fixed) is said to be feasible if we can slide the unit squares around to get back to the picture above.
Do: Prove that exactly half of the possible permutations in Sam Lloyd's Puzzle

are feasible. The transposition of 1 and 2 is not feasible.
Do: Define feasibility for a Rubik's Cube in the same way, but replace the squares with the "cubies" of Rubik's Cube. Prove that switching two corner cubies is not feasible.
Homework 7.1: Prove that the n-cycle $(12\dots n)$ and the transposition (12) generate S_n.
Puzzle: Prove that fewer than $n-1$ transpositions cannot generate S_n. (This has an ah-ha solution.)
Puzzle*: Prove that the number of $(n-1)$-tuples of transpostions that generate S_n is n^{n-2}. (There is no ah-ha solution.)
Diagonal, upper triangular and lower triangular matrices and their determinants.
Definition. A matrix $A \in F^{n\times n}$ is *non-singular* if its columns are linearly independent.
Theorem. A matrix A is non-singular if $\det(A) \neq 0$.
Theorem. The following are equivalent:
(a) A is non-singular.
(b) $\text{rk}(A) = n$.
(c) The rows of A are linearly independent.
(d) The columns of A form a basis for F^n.
(e) The rows of A form a basis for F^n.
(f) $\det(A) \neq 0$.
(g) $\exists A^{-1}$.
Matrix product: For $A = (a_{ij})_{ij} \in F^{k\times\ell}$ and $B = (b_{ij})_{ij} \in F^{\ell\times m}$, $AB = (c_{ij})_{ij} \in F^{k\times m}$ is defined by

$$c_{ij} = \sum_{t=1}^{\ell} a_{it}b_{tj},$$

Do: $\text{rk}(AB) \leq \min\{\text{rk}(A), \text{rk}(B)\}$.
Do: Prove $(AB)C = A(BC)$.
Definition. The *trace* of a matrix $A = (a_{ij})_{n\times nj}$ is $\text{Tr}(A) = \sum_{i=1}^{n} a_{ii}$.
Homework 7.2: Prove if $A \in F^{k\times\ell}$ and $B \in F^{\ell\times k}$ then $\text{Tr}(AB) = \text{Tr}(BA)$.
Puzzle: If $A, B \in F^{n\times n}$ prove $\det(AB) = \det(A)\det(B)$.
Do: Prove $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$.
Do: Show that $\text{rk}(A) = 0$ if and only if $A = 0$, the zero matrix.
Do: Let I_n be the $n \times n$ identity matrix and J_n be the $n \times n$ "all-ones matrix" (for which every entry is 1). Find a formula for $\det(J_n - I_n)$. (It's a very simple expression.)
Do: Suppose $A = (a_{ij})_{n\times n}$ and $a_{ii} = a$, $a_{ij} = b$ for $j > i$, and $a_{ij} = c$ for $j < i$. Compute $\det(A)$. (Again, there is a simple expression.)
Do: Suppose $A = (a_{ij})_{n\times n}$ and $a_{ij} = 1$ if $i - 1 \leq j \leq i + 1$, and $a_{ij} = 0$ otherwise. Compute $\det(A)$.
An *inverse* of a matrix $A \in F^{n\times n}$ is a matrix B such that $AB = I_n$, if such a matrix exists we write $B = A^{-1}$ and say A is *invertible.*
Do: Show that A^{-1} exists if and only if A is non-singular. (Gaussian elimination.)
Do: Show that if A is invertible then $A^{-1}A = I_n$. That is, given A, if $AB = I_n$ then $BA = I_n$.