Advanced Combinatorics
Math 295, Spring 2007
Class Summary and Homework Questions: Week 2, part 3

(Proofread by the instructor.)

Lecture 8 - April 5th

The Homework below is due on Wednesday, March 11. Yesterday's homework is due on Tuesday, March 10.

The return to Oddtown and rank inequalities:
$\mathrm{rk}(AB) \leq \min\{\mathrm{rk}(A), \mathrm{rk}(B)\}$.
Transpose of the $k \times \ell$ matrix $A = (a_{ij})$ is the $\ell \times k$ matrix $A^T = (a_{ji})$.
Do: Prove for $A \in F^{n\times\ell}$ and $B \in F^{\ell\times k}$ that $(AB)^T = B^T A^T$.
Do: Prove $\mathrm{rk}(AB) \leq \mathrm{rk}(B)$ directly without use of the transpose and the result $\mathrm{rk}(AB) \leq \mathrm{rk}(A)$.
Do: Find 2×2 matrices A and B such that $AB \neq 0$ but $BA = 0$.
Do: Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that $A \neq 0$ but $A^2 = 0$.
Do: Prove if $A \in F^{n\times n}$ and $A^k = 0$, then $A^n = 0$.
Theorem (Systems of homogeneous linear equations.) For $A \in F^{m\times n}$, let $U = \{\mathbf{x} \in F^n \mid A\mathbf{x} = \mathbf{0}\}$ be the set of solution to the homogeneous system of linear equations with matrix A. Then $U \leq F^n$ (subspace), and

$$\dim(U) = n - \mathrm{rk}(A).$$

Do: Prove the formula above using Gaussian Elimination. Hint: Use elementary row operations and column permutations to put A in the form

$$\left(\begin{array}{cccc|ccc}
\lambda_1 & 0 & \cdots & 0 & * & \cdots & * \\
0 & \lambda_1 & \cdots & 0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & \lambda_r & * & \cdots & * \\
\hline
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right)$$

where $\lambda_i \neq 0$ and r is the rank of A. (You need to justify why we can perform these operations on A.)
Observe that (after the permutation leading to the above form), we can choose the values of the last $n-r$ unknowns arbitrarily; those values uniquely determine the values of the the first r variables. To obtain a basis of the subspace of solutions, we take the standard basis of $\mathbb{R}^{n-r}$ for the last $n-r$ variables and extend it in the unique way to the first r variables.
For $\mathbf{a}, \mathbf{b} \in F^n$ we say $\mathbf{a}$ is perpendicular to $\mathbf{b}$ and write $\mathbf{a} \perp \mathbf{b}$ if $\mathbf{a} \cdot \mathbf{b} = 0$. For $T \subseteq F^n$, we set $T^\perp = \{\mathbf{v} \in F^n \mid \mathbf{v} \cdot \mathbf{t} = 0 \text{ for all } \mathbf{t} \in T\}$.
Do: Show $T^\perp \leq F^n$ (subspace).
Theorem. $\dim(T^\perp) = n - \mathrm{rk}(T)$.
Do: $T^\perp = (\mathrm{span}(T))^\perp$.
Theorem. If $U \leq F^n$, then $\dim(U) + \dim(U^\perp) = n$.
An element $\mathbf{v} \in F^n$ is *isotropic* if $\mathbf{v} \neq 0$ but $\mathbf{v} \cdot \mathbf{v} = 0$. A subspace $U \leq F^n$ is

totally isotropic if $U \perp U$, i. e., for all $\mathbf{x}, \mathbf{y} \in U$, $\mathbf{x} \perp \mathbf{y}$, i. e., $U \leq U^{\perp}$.
Corollary. If $U \leq F^n$ is totally isotropic then $\dim(U) \leq n/2$.
Do: Show that if $U \leq F^n$ and $\dim(U) = d$ then there exists a basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$ of U and every vector in U is uniquely expressible as a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$.
Do: A d-dimensional subaspace of $\mathbb{F}_p^n$ has p^d elements.
Eventown Theorem: a maximal set of Eventown clubs is a totally isotropic subspace of $\mathbb{F}_2^n$; therefore Eventown has at most $2^{\lfloor n/2 \rfloor}$ clubs.
Homework 8.1: (10 points) In F^{2n} construct totally isotropic subspaces of dimension n for $F = \mathbb{C}$ and $F = \mathbb{F}_5$.
Homework 8.2: (10 points) Consider the variant of the Oddtown problem where all the $|C_i|$ are even and $|C_i \cap C_j|$ are odd. Prove $m \leq n + 1$.
Homework 8.3: (10 points) Suppose $\alpha_1, \dots, \alpha_n \in F$ and $\alpha_i \neq \alpha_j$ for $i \neq j$, and set $f(x) = \prod_{i=1}^{n}(x - \alpha_i)$ and $g_j(x) = f(x)/(x - \alpha_j)$. Prove that the polynomials $g_1, \dots, g_n$ are linearly independent over F.
Homework 8.4: (20 points) Suppose $X_1, \dots, X_n$ are pairwise independent non-constant random variables on (Ω, P). Prove $|\Omega| \geq n+1$. Hint: (1) Establish that we may assume $E(X_i) = 0$ without loss of generality. (2) Establish that the random variables over (Ω, P) form a vector space of dimension $|\Omega|$ over $\mathbb{R}$. (3) Use $E(XY)$ as an "inner product" to prove that under assumption (1), the variables $1, X_1, \dots, X_n$ are linearly independent over $\mathbb{R}$.
Reading: In the blue text read 1.1 (Oddtown) and 2.3 (orthogonality and Eventown). Background can be found in 2.1 and 2.2.