Advanced Combinatorics Math-25905 Quiz Zero. March 26, 2007
Instructor: László Babai

Name: _____

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided. (You may continue on the reverse.) Indicate the *proof* of your statements except where explicitly requested not to. If you are not sure you understand a problem properly, **ask the instructor.** This quiz serves as a "placement test" only (to inform the instructor of your background) and has no grade value.

1. (a) Define what it means that a sequence $\{a_n\}_{n=1}^{\infty}$ converges to the finite limit L. Watch your quantifiers ("for all," "there exists" statements). If you know the $\forall, \exists$ notation, use it and avoid any English words in the definition. (b) Give an example of a sequence that is bounded yet it does not converge to any limit.

2. (a) Define the rank of a matrix. (b) Let A and B be two $k \times n$ matrices over the real numbers. Prove: $\mathrm{rank}(A+B) \le \mathrm{rank}(A) + \mathrm{rank}(B)$. (c) Let A be a $k \times n$ integral matrix (a matrix with integer entries). Prove: $\mathrm{rank}(A) \ge \mathrm{rank}_2(A)$ where rank_2 referes to the mod-2-rank, i. e., the rank over the field of order 2. (d) Give a 3×3 $(0,1)$ matrix (all entries 0 or 1) where the inequality in part (c) is strict.

3. (a) What is the probability that out of n coin flips, exactly k come up heads? (b) What is the probability of a flush in poker? A poker hand is 5 cards out of the standard deck of 52 cards. A poker-hand is a *flush* if all cards are of the same suit (all of them are spades, or all are hearts, or all are diamonds, or all are clubs). There are 13 cards of each suit in the standard deck. Give a simple expression, do not evaluate. (c) What is the probability that no two cards in a poker hand are of the same kind (i. e., there are no two Kings, no two 9s, etc, in the hand)? There are 13 kinds of cards, four of each kind, in the standard deck. Give a simple expression, do not evaluate. (e) What is the expected number of Kings in a poker hand?

4. Give very simple (one-line) proofs of the inequalities (a) $n! > (n/\mathrm{e})^n$ (do not use Stirling's formula); (b) $\binom{n}{k} < (\mathrm{e}n/k)^k$.

5. A row of 100 coins is placed on a table. Alice and Bob take turns pocketing a coin from one of the ends (they choose each time, which end). Prove: Alice can ensure that she gets at least as much money as Bob. Example with 4 coins: 10 25 5 1. Alice, by modestly starting with the penny, ensures a total of 26 cents against Bob's 15 (Alice: 1, Bob: 10, Alice 25, Bob: 5). Had Alice been greedy and started with the dime, she would have ended up with 15 cents against Bob's 26 (Alice: 10, Bob: 25, Alice: 5, Bob: 1).