Advanced Combinatorics Math-25905 Quiz One. March 29, 2007
Instructor: László Babai

Name: ____

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided. (You may continue on the reverse.) Indicate the *proof* of your statements except where explicitly requested not to. If you are not sure you understand a problem properly, **ask the instructor.** This quiz contributes 15% to your course grade.

1. (8+8 points) (a) Define what it means for two sequences $\{a_n\}$ and $\{b_n\}$ to be in the Θ relation: $a_n = \Theta(b_n)$. Do not use the big-Oh and big-Omega notation in your definition; use no English words except for logical connectives ("AND," "IF ... THEN," etc.). (b) Give an example of two sequences $a_n > 0$ and $b_n > 0$ such that $a_n = \Theta(b_n)$ but the limit $\lim_{n\to\infty} a_n/b_n$ does not exist.

2. (10+8+4+(3+5)) Give very simple (one-line) proofs of the inequalities (a) $n! > (n/\mathrm{e})^n$ (do not use Stirling's formula) ($n \geq 1$); (b) $\binom{n}{k} < (\mathrm{e}n/k)^k$ ($1 \leq k \leq n$). (c) State Stirling's formula (use asymptotic notation). (d) Does the inequality under (a) follow from Stirling's formula (d1) for all n? (d2) for all sufficiently large n? Indicate, why.

3. (5+8+5+10 points) (a) Define the prime counting function $\pi(x)$. (b) State the Prime Number Theorem. (c) Name the two mathematicians who proved it. (d) True or false: $\pi(x) = \Theta(x/\log_{10} x)$. Prove your answer.

4. (30 + 8 points) (a) Prove: if a prime power p^t divides the binomial coefficient $\binom{n}{k}$ then $p^t \le n$. State and prove any lemmas you use. (b) State the part of a major theorem we proved using (a).

5. (10+8 points) Let $d(n)$ denote the number of positive divisors of the positive integer n. Prove: $d(n)$ is odd if and only if n is the square of an integer. (This is a puzzle problem, the proof is 2 lines. You get the +8 points if your proof is "from scratch:" it does not use any results stated but not proved in class.)

6. (20 points) A row of 100 coins is placed on a table. Alice and Bob take turns pocketing a coin from one of the ends (they choose each time, which end). Prove: Alice (the first player) can ensure that she gets at least as much money as Bob. Example with 4 coins: 10 25 5 1. Alice, by modestly starting with the penny, ensures a total of 26 cents against Bob's 15 (Alice: 1, Bob: 10, Alice 25, Bob: 5). (Hint: Alice's strategy is very simple, takes two lines to describe.)