Advanced Combinatorics Math-25905 Quiz Two. April 5, 2007
Instructor: László Babai

Name: ________________________________

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided. (You may continue on the reverse.) Indicate the *proof* of your statements except where explicitly requested not to. If you are not sure you understand a problem properly, **ask the instructor.** This quiz contributes 15% to your course grade.

1. (16 points) Consider the vectors $a_1 = (1, 1, 0, 0, 0)$, $a_2 = (0, 1, 1, 0, 0)$, $a_3 = (0, 0, 1, 1, 0)$, $a_4 = (0, 0, 0, 1, 1)$, $a_5 = (1, 0, 0, 0, 1)$. Prove that (a) they are linearly independent over $\mathbb{R}$; (b) they are linearly independent over F_p if and only if $p \neq 2$.

2. (10 points) In a string of n coin flips, what is the expected number runs of k heads? (A "run of k heads" is a sequence of k consecutive heads: so for instance the string $HTHHHHHHT$ has 3 runs of four heads.) Prove your answer. Give an exact definitions of any random variables you use; the clarity of this definition accounts for half the credit.

3. (1+2+6+14 points) (a) Define the prime counting function $\pi(x)$. (b) State the Prime Number Theorem. (c) Name the two mathematicians who proved it. (d) True or false: (d1) $\pi(x) = O(x^{0.9})$; (d2) $\pi(x) = \Omega(x^{0.9})$. Prove your answers.

4. (18 points) Let $d(n)$ denote the number of positive divisors of the positive integer n. Prove: $d(n) < 2\sqrt{n}$.

5. (14+18 points) (a) Suppose $k_n = o(\sqrt{n})$. Prove that $\binom{n}{k_n} \sim \frac{n^{k_n}}{k_n!}$.
(b) Suppose $\sqrt{n} = o(\ell_n)$. Prove that $\binom{n}{\ell_n} = o\left(\frac{n^{\ell_n}}{\ell_n!}\right)$.

6. (28 points) Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be subsets of an n-set X. Assume that
(a) $(\forall i)(\,|A_i \cap B_i|$ is odd$\,)$;
(b) $(\forall i > j)(\,|A_i \cap B_j|$ is even$\,)$.
Prove: $m \le n$.

7. (3+12+8 points) We roll n dice. (Each die shows an integer between 1 and 6.) Let X be the sum and Y the product of the numbers shown. Determine (a) $E(X)$; (b) $\mathrm{Var}(X)$; (c) $E(Y)$.

8. (Bonus problem, 20 points) ("Cleaning the corner") We label the cells of the positive quadrant (the "game board") by pairs of integers $\{(i, j) : i, j \geq 0\}$. The neighbor to the North of cell (i, j) is cell $(i + 1, j)$; the neighbor to the East is cell $(i, j + 1)$. The corner cell is $(0, 0)$. The Manhattan distance between cells (i_1, j_1) and (i_2, j_2) is $|i_1 - i_2| + |j_1 - j_2|$.

   Chips are placed on some of the cells, at most one chip per cell. Chips "shift and multiply" in the following manner: suppose a chip is on cell (i, j). If both its neighbor to the North and its neighbor to the East are empty, we can remove the chip from (i, j) and place a chip on its neighbor to the North and another chip on the neighbor to the East.

   Initially we put a chip on cell $(0, 0)$; otherwise the game board is empty. We wish to clean the corner, i. e., we wish to achieve, by a sequence of "shift/multiply" moves, that there be no chip left within Manhattan distance d from the corner. Prove that this is impossible (a) for $d = 3$; (b) for $d = 2$.