Advanced Combinatorics Math-25905 Quiz Three. April 13, 2007
Instructor: László Babai

Name: ______________________________________

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided. (You may continue on the reverse.) Indicate the *proof* of your statements except where explicitly requested not to. If you are not sure you understand a problem properly, **ask the instructor.** WARNING: The bonus problems are underrated; work on them only after you are done with the regular problems. This quiz contributes 20% to your course grade.

1. (5+9 points) (a) Define the relation $a_n = \Theta(b_n)$ between the sequences $\{a_n\}$ and $\{b_n\}$. Do not use the big-Oh and big-Omega notations; your definitions should be a properly quantified formula with no English words except for logical connectives. (b) Prove: if $a_n, b_n \to \infty$ and $a_n = \Theta(b_n)$ then $\ln a_n \sim \ln b_n$.

2. (6+6+2+14 points) (a) Given a finite probability space (Ω, P), define what is a random variable. (b) Given a vector space V over a field F, define what is a basis. (c1) Observe that the random variables over the finite probability space (Ω, P) form a vector space under the natural notion of linear combinations. Do not prove this fact, but name the field of scalars. (c2) Determine the dimension of this space by giving a basis. Prove that what you found is indeed a basis.

3. (12 points) Let F be a field. Let $f_1, \dots, f_m$ be polynomials over F (in one variable) and let $\alpha_1, \dots, \alpha_m \in F$. Consider the $m \times m$ matrix $A = (f_i(\alpha_j))$. Prove: if $\det(A) \neq 0$ then the polynomials $f_1, \dots, f_m$ are linearly independent.

4. (12+4+4+4+15 points) "Divisor-town" has n citizens, numbered 1 through n. They form n clubs, $C_1, \dots, C_n$. Citizen i belongs to club C_j if $i \mid j$. (Example: the members of club C_6 are citizens 1,2,3, and 6.) (a) Prove that the incidence vectors of the clubs are linearly independent. (b) Let $t(n)$ denote the number of clubs with exactly 2 members. Asymptotically evaluate $t(n)$ by stating the relevant theorem stated in class. Do not prove. (c) What is the number of clubs to which citizen i belongs? (Give the exact number.) (d) What is the number of members of club C_j? (Give a clear English description and common notation we used; do not evaluate.) (e) Prove that the average size of the clubs in Divisor-town is asymptotically equal to $\ln n$. Hint: first prove that the average size of the clubs is the same as the average number of clubs to which a citizen belongs.

5. (14 points) Let $\{\ell_n\}$ be a sequence of positive integers. Suppose $\sqrt{n} = o(\ell_n)$ (little-oh). Prove that $\binom{n}{\ell_n} = o\left(\frac{n^{\ell_n}}{\ell_n!}\right)$.

6. (5+20+20 points) SOLVE ONLY ONE OF THE FOLLOWING PROBLEMS, (A) OR (B). If you solve both, you only get credit for the better solution. Those who have done well are expected to choose (A).
(A) A $k \times \ell$ *submatrix* of an $n \times m$ matrix is obtained by selecting k rows and ℓ columns and keeping only those cells where the selected rows intersect the selected columns. (Aa) What is the number of $k \times \ell$ submatrices of an $n \times m$ matrix? (Ab) A $(0,1)$-matrix is a matrix all elements of which are 0 or 1. A matrix is *homogeneous* if all of its elements are the same. Let $n \rightsquigarrow k$ denote the statement that "every $n \times n$ $(0,1)$-matrix contains a $k \times k$ homogeneous submatrix." Prove: $\lfloor 2^{k/2} \rfloor \not\rightsquigarrow k+1$. (Hint: probabilistic method.) (Ac) Prove: $2k4^k \rightsquigarrow k$. Hint: prove, by induction on j, that every $n \times j$ $(0,1)$-matrix contains a $\left\lceil \frac{n}{2^j} \right\rceil \times j$ submatrix all of whose columns are homogeneous.
(B) (Ba) Define the Erdős-Rado arrow notation $n \to (k, \ell)$. (Bb) Prove: $\lfloor 2^{k/2} \rfloor \not\to (k+1, k+1)$. (Bc) Prove: for all sufficiently large k we have $4^k \to (k+100, k+100)$. Estimate how large a value of k is large enough for your proof to work. If you can't prove this, prove $4^k \to (k+1, k+1)$ for 2/3 of the credit.

7. (14 points) Prove: if we have an explicit construction that proves that for every prime p and every $n > p^2$ we have

$$\binom{n}{p^2-1} \not\to \left(\binom{n}{p-1}+1, \binom{n}{p-1}+1\right)$$

then it follows that the same explicit construction proves $(\forall C)(\exists N_0)(\forall N > N_0)(N^C \not\to (N, N))$.

8. (14 points) Let $A_1, \ldots, A_m$ be subsets of $\{1, \ldots, n\}$. Assume that
(i) $(\forall i)(|A_i| \in \{26, 29\})$, and
(ii) $(\forall i \neq j)(|A_i \cap A_j| \in \{0, 4, 6, 10, 18\})$.
Prove: $m \leq \binom{n+1}{2}$. You may refer to the third week class notes and use a theorem stated there. Name the theorem you use.

9. (20 points) Prove the lower bound part of Chebyshev's weak version of the Prime Number Theorem: $\pi(x) = \Omega(x/\ln x)$. Use without proof the main lemma which says that if a prime power p^t divides the binomial coefficient $\binom{n}{k}$ then $p^t \leq n$.

10. (BONUS PROBLEM, 15 points) Prove: every nonzero polynomial has a nonzero multiple with only prime exponents. In other words, given $f \in F[x]$ such that $f \neq 0$, there exists $g \in F[x]$ such that $g \neq 0$ and $fg = a_2x^2 + a_3x^3 + a_5x^5 + \dots$.

11. (BONUS PROBLEM, 3+15 points) Consider the statement that "the probability that two random positive integers are relatively prime is c. (a) Make sense out of this statement (define). (b) Assuming the value c, prove that its value must be $c = 6/\pi^2$. Use the fact that this number is the reciprocal of $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ (Euler).