SOLUTIONS: HOMEWORK 1.1 (MARCH 26-28, 2007)

Based on John Sternberg's solution.

1.1 Suppose $a_n, b_n > 1$. Consider the statements (1) $a_n \sim b_n$ and (2) $\ln a_n \sim \ln b_n$. Show
(a) that (1) does not imply (2), and
(b) if, in fact, $a_n > 1 + \varepsilon$ for some fixed positive ε then (1) does imply (2).
Solution

($1 \not\Rightarrow 2$) Let $a_n = \mathrm{e}^{1/n}$ and $b_n = \mathrm{e}^{1/n^2}$. Since $1/n \to 0$, by the continuity of e^x at $x = 0$ we have that $a_n \to \mathrm{e}^0 = 1$. Similarly, $b_n \to 1$ and therefore $a_n/b_n \to 1$, i. e., $a_n \sim b_n$ as desired. On the other hand,

$$\frac{\ln a_n}{\ln b_n} = \frac{1/n}{1/n^2} = n \to \infty$$

and therefore $\ln a_n \not\sim \ln b_n$.

($1 \Rightarrow 2$) Let $c = \ln(1 + \varepsilon)$. Then c is a positive constant and $\ln a_n > c$.
Now $b_n/a_n \to 1$ and therefore $\ln b_n - \ln a_n = \ln(b_n/a_n) \to 0$. Observe that

$$\frac{\ln(b_n)}{\ln(a_n)} = 1 + \frac{\ln(b_n) - \ln(a_n)}{\ln(a_n)}.$$

Now the numerator of the fraction on the right hand side goes to 0 while the denominator stays above the positive constant c, therefore the quotient goes to 0 and the entire right hand side goes to 1. In other words, $\ln b_n \sim \ln a_n$, as desired.

The solutions to Homework problems 1.2 and 1.3 have been discussed in detail in class. The answer to 1.4 (number of permutations with a single cycle) is $(n-1)!$ (start at a point, choose one of the remaining $n-1$ as its successor along the cycle, followed by one of the remaining $(n-2)$, etc.).