SOLUTIONS: HOMEWORK 2 (MARCH 27-29, 2007)

Solutions by John Sternberg, handed in in Latex and slightly modified by the instructor.

2.1 Prove the inequality

$$\binom{2k+1}{k} < 4^k.$$

Solution. Recall that

$$2^{2k+1} = \binom{2k+1}{0} + \binom{2k+1}{1} + \cdots + \binom{2k+1}{k} + \binom{2k+1}{k+1} + \cdots + \binom{2k+1}{2k+1}.$$

By the symmetry of the binomial coefficients, we have $\binom{2k+1}{k} = \binom{2k+1}{k+1}$. Therefore,

$$\binom{2k+1}{k} + \binom{2k+1}{k+1} < 2^{2k+1}$$
$$2 \cdot \binom{2k+1}{k} < 2^{2k+1}$$
$$\binom{2k+1}{k} < 2^{2k} = 4^k.$$

2.2 Show $k^2 \nrightarrow (k+1, k+1)$.

Solution. Consider a set of k^2 vertices, and organize them into k "blocks" of k vertices each. Make all possible connections within each block red. So we have k disjoint all-red blocks. Now make all other connections in blue. We claim that the resulting coloring has no homogeneous $(k+1)$-tuples.

Indeed, any set of $k+1$ vertices contains vertices from at least two blocks and therefore must have a blue connection. On the other hand, by the Pigeon Hole Principle, any set of $k+1$ vertices must have at least two vertices in the same block and therefore must have a red connection. We conclude that any set of $k+1$ vertices has connections of both colors and is therefore not homogeneous.

2.3 (a) Extend the Erdős-Rado arrow notation to give meaning to $n \to (k, \ell, m)$.

(b) Prove $17 \to (3, 3, 3)$.

Solution. (a) We shall say that $n \to (k, \ell, m)$ if no matter how we color the $\binom{n}{2}$ pairs of n objects red, blue, and green, there will be either an all-red k-subset, or an all-blue ℓ-subset, or an all-green m-subset.

(b) Given a coloring of the pairs of 17 objects ("vertices"), we need to show that it has a triangle in one of the colors. Let us fix a vertex, v. It has 16 connections extending from it. At least six of these connections must be of one color, because otherwise we would have at most $5 + 5 + 5 = 15$ vertices other than v (this, too, is a case of the PHP). So choose a set R of six vertices connected to v by the same color, say red. If any two vertices within R are connected by a red edge, these two vertices plus v form a red triangle. So if there is no red triangle, then all connections within R are blue or green. But we know that $6 \to (3, 3)$, so within R we shall have either a blue or a red triangle.