Combinatorics CS-284/Math-274 First Midterm. April 30, 2008
Instructor: László Babai

Show all your work. **Do not use book, notes, or scrap paper.** Warning: the bonus problems are underrated; solve the regular problems first. This midterm contributes 21% to your course grade.

1. (15 points) Prove Birkhoff's Theorem: Every doubly stochastic matrix is a convex combination of permutation matrices. (Recall that a doubly stochastic matrix has nonnegative entries and every row sum is 1 and every column sum is 1.)

2. (2+8 points) (a) Define Steiner Triple Systems (STS). (b) Prove: if a STS has n points then a proper sub-STS has at most $(n-1)/2$ points.

3. (2+6 points) (a) Define orthogonl Latin Squares. (b) If $n \geq 3$ is odd, construct two orthogonal $n \times n$ Latin Squares. Prove that they are orthogonal. Highlight the point in your proof where you use that n is odd.

4. (6+1+1 points) (a) Draw a rather large image of the Fano plane, and put homogeneous coordinates on each vertex (e. g. 001) and on each line. (b) What is the field over which your homogeneous coordinates are defind? (c) If the homogeneous coordinates of a point p are $[x, y, z]$ and the homogeneous coordinates of a line ℓ are $[a, b, c]$, what is the condition these six parameters must satisfy to indicate that p and ℓ are incident?

5. (4+1+8 points) (a) For a hypergraph $\mathcal{G} = (V, \mathcal{E})$ where $\mathcal{E} = \{E_1, \ldots, E_m\}$ is the set of edges, define the parameters τ (covering number, a. k. a. "hitting number"), ν (matching number), and their linear relaxations τ^* and ν^*. (b) Let $\mathcal{G}$ be a finite projective plane of order n. View $\mathcal{G}$ as a hypergraph. Note that it is uniform; what is the size of each edge? Note also that it is regular; of what degree? (c) Determine the values ν, ν^*, τ, τ^* for this hypergraph. Prove your answers.

6. (1+8 points) Consider a random graph $\mathcal{G}$. The set of vertices is $[n] = \{1, \ldots, n\}$; and $\mathcal{G}$ is chosen uniformly from all graphs on this set of vertices. (a) What is the size of the sample space (number of possible outcomes of this experiment)? (b) What is the expected number of triangles in $\mathcal{G}$? Prove your answer. Make sure you give a clear definition of each random variable used; do not confuse events with random variables.

(OVER)

7. (BONUS 5 points) Let $\mathcal{G} = (P, L, I)$ be a finite projective plane of order n. An *oval* in $\mathcal{G}$ is a subset $T \subseteq P$ such that no three points is T are on a line. Prove: $|T| \leq n + 2$.

8. (BONUS 5 points) Prove: all sufficiently large finite projectiv planes are 2-colorable (the points can be colored red and blue such that none of the lines becomes monochromatic).