Combinatorics CS-284/Math-274 Second Midterm. May 14, 2008

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Show all your work. **Do not use book, notes, or scrap paper.** Warning: the bonus problems are underrated; solve the regular problems first. This midterm contributes 21% to your course grade.

1. (4+8 points) (a) For a hypergraph $\mathcal{G} = (V, \mathcal{E})$ where $\mathcal{E} = \{E_1, \dots, E_m\}$ is the set of edges, define the parameters τ (covering number, a. k. a. "hitting number"), ν (matching number), τ^* (fractional covering number), and ν^* (fractional matching number). (b) Let $\mathcal{G}$ be a finite projective plane of order n. View $\mathcal{G}$ as a hypergraph. Determine the values ν, ν^*, τ, τ^* for this hypergraph. Prove your answers.

2. (14 points) Prove: $\text{ex}(n, K_{2,3}) = O(n^{3/2})$. In other words, if a graph with n vertices and m edges does not contain the complete bipartite graph $K_{2,3}$ as a subgraph then $m = O(n^{3/2})$.

3. (13+1 points) (a) Prove: $\text{ex}(n, K_{2,2}) = \Omega(n^{3/2})$. (b) Infer from (a) that $\text{ex}(n, K_{2,3}) = \Omega(n^{3/2})$.

4. (1+1+4 points) (a) State Jensen's inequality about convex functions. (b) State the inequality between the arithmetic and geometric means of n positive numbers. (Define these quantities.) (c) Use (a) to prove (b). To what function will you need to apply Jensen's inequality?

5. (1+4+3+3 points) (a) The statement $17 \to (3,3,3)$ involves the Erdős-Rado arrow symbol. Explain what this statement means. (b) Prove that $17 \to (3,3,3)$. (c) Let r_t denote the smallest value such that $r_t \to (3,3,\dots,3)$ (t times). Prove: $r_t \le t r_{t-1} - t + 2$. (d) Infer from (c) that $r_t \le 1 + t!e$.

6. (1+5 points) Consider a random graph $\mathcal{G}$. The set of vertices is $[n] = \{1, \dots, n\}$; and $\mathcal{G}$ is chosen uniformly from all graphs on this set of vertices. (a) What is the size of the sample space (number of possible outcomes of this experiment)? (b) What is the expected number of Hamilton cycles in $\mathcal{G}$? (Recall: a Hamilton cycle is a cycle that passes through all vertices.) Prove your answer. Make sure you give a clear definition of each random variable used; do not confuse events with random variables.

7. (BONUS 5 points) Prove: if a graph on n vertices has no triangle then its chromatic number is $O(\sqrt{n})$.

(OVER)

8. (BONUS 1+1+8 points) Let G be a graph with n vertices and m edges. (a) Define the matching polytope of G. This is a convex set in the space of what dimension (over $\mathbb{R}$)? (b) State a set of $n + m + 2^{n-1}$ linear constraints satisfied by all points of the matching polytope. This set of constraints should also suffice to define the matching polytope but you don't need to prove this (Edmonds' Theorem). (c) Prove: if G is bipartite then the matching polytope is defined by the first $n + m$ of these constraints. (Do not assume any result we did not prove in class; in particular, do not assume Edmonds' Theorem.) Hint: it suffices to prove that every vertex of the polytope defined by your set of $m + n$ constraints is integral.