Combinatorics CS-284/Math-274/CS-372 First Quiz. April 11, 2008
Instructor: László Babai

Name: _____ **U** **G** (circle one)

Show all your work. **Do not use book, notes, or scrap paper.** Write your answers in the space provided. You may continue on the reverse. When describing an algorithm in pseudocode, **explain the meaning of your variables** (in English). Notation: **U**: "undergraduate problem" (required). **G**: "graduate problem" (required only if you seek graduate (CS-372) credit; those seeking undergraduate (CS-284 or Math-274) credit should view these as bonus problems for the stated number of points; **B**: bonus problem for all. Warning: the bonus problems are underrated; solve the regular problems first. – This quiz contributes 8% to your course grade.

1. **(U)** (2+5 points) Recall that for a graph G, we use τ to denote the minimum cover (smallest number of vertices that hit every edge) and ν to denote the maximum matching size (maximum number of disjoint edges). We proved that $\tau \leq 2\nu$. (a) Prove that this inequality is tight for all ν. (b) Make your examples connected (for all ν). (Clearly define your examples and state their values of τ and ν in terms of n, the number of vertices. Do not prove. A picture helps. - A correct solution to (b) earns you the 2 points for (a) as well.)

2. **(U)** (2+5 points) (a) Define Steiner Triple Systems (STS). (b) Prove that if a Steiner triple system has n points then $n \equiv 1$ or 3 mod 6.

3. **(U)** (5 points) Give an example of a probability space and three events in it that are pairwise independent but not fully independent. Make your sample space as small as possible. Prove the stated facts about independence but do not prove that the size is smallest possible.

4. (6+5 points) (a) **(B)** Prove: if a bipartite graph has a perfect matching then it has a vertex x such that every edge out of x belongs to a perfect matching. (b) **(U)** Use part (a) to prove that a regular bipartite graph of degree r has at least $r!$ perfect matchings.

5. **(G)** (6 points) Beyond the seven seas there is a tiny island, 6 square miles in all. The island is inhabited by six native tribes and by six turtle species. Each tribe and each turtle species occupies one square mile of territory; the territories of the tribes don't overlap with one another; nor do the territories of the different turtle species.

   Each tribe wishes to select a totem animal from among the turtle species found in the tribe's territory; and each tribe must have a different totem animal. Prove that such a selection is always possible. State the exact result proved in class that you use.