CMSC-37110 Discrete Mathematics
FINAL EXAM December 8, 2008

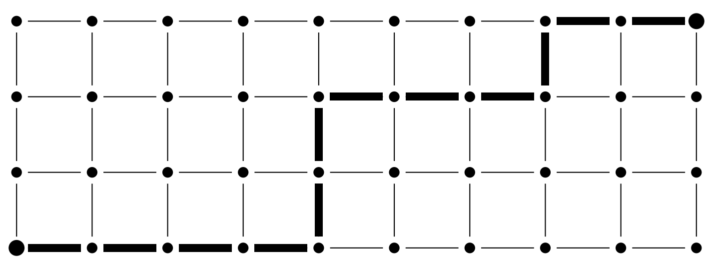
This exam contributes 32% to your course grade.

Do **not** *use book, notes.* You **may** use a calculator for *basic arithmetic* but not for more advanced functions such as g.c.d. or determinant. **Show all your work.** If you are not sure of the meaning of a problem, **ask the proctor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (21 points) Let p be an odd prime and $k \not\equiv 0 \pmod{p}$. Prove:

$$k^{(p-1)/2} \equiv \pm 1 \pmod{p}.$$

2. (15 points) Count the shortest paths from the bottom left corner to the top right corner of the $n \times k$ grid. (Note that the length of such a path is $n + k - 2$). Your answer should be a very simple formula. Prove your answer. (The figure shows a 4×10 grid with a shortest path highlighted.)



3. (15 points) In a well-shuffled deck of n cards, numbered 1 through n, what is the probability that cards #1 and #2 come next to each other (in either order)? Your answer should be an extremely simple expression; make it as simple as possible. Prove your answer.

4. (24 points) Calculate the determinant of the $n \times n$ matrix $A_n = (a_{i,j})$ where $a_{i,i} = 1$ ($i = 1, \ldots, n$), $a_{i,i+1} = -1$ ($i = 1, \ldots, n-1$), and $a_{i,i-1} = 1$ ($i = 2, \ldots, n$), all other entries are zero. The figure shows the matrix A_5. Hint: Let $d_n = \det(A_n)$. Experiment with small n; observe the pattern, make a conjecture. To prove the conjecture, expand by the last row to obtain a recurrence for the sequence d_n.

$$A_5 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

5. (10+8+8B points)

   (a) Let $\{a_n\}$ be a sequence of real numbers. Define, with a well-quantified formula, that $a_n \neq O(1)$. The formula should begin with quantifiers; no Boolean operations (negation, and, or, if-then) should precede any quantifier. No English words permitted.

   (b) Does $a_n \neq O(1)$ mean $|a_n| \to \infty$?

   (c) (BONUS) Let $f(n)$ denote the number of solutions to the equation $x+y=n$ where x and y are prime numbers. For instance, $f(10) = 3$ (the solutions are $3+7, 5+5, 7+3$). Prove that $f(n) \neq O(1)$.

6. (12 points) Let a, b, d be integers. Suppose (a) d is a common divisor of a and b and (b) d is a linear combination of a and b (with integer coefficients). Prove: d is a g.c.d. of a and b. State clearly what it is that you need to prove according to the definition of g.c.d.

7. (15 points) Find an $n \times n$ matrix of rank one such that all the n^2 entries are distinct integers.

8. (10 points) Let V be an inner product space. Suppose the vectors $v_1, \dots, v_k \in V$ are orthonormal, i. e., $\langle v_i, v_j \rangle = \delta_{i,j}$. Prove that $v_1, \dots, v_k$ are linearly independent.

9. (4+20+10+8B points)

   (a) When do we say that the $n \times n$ matrices C, D are similar?

   (b) Recall that an $n \times n$ matrix is *diagonalizable* if it is similar to a diagonal matrix. Consider the following two matrices:
   $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

      (b1) One of these is diagonalizable as an immediate consequence of a major theorem (no calculation needed). Which matrix? State the theorem. Calculate the diagonal matrix to which your matrix is similar.

      (b2) Prove that the other matrix is not diagonalizable.

   (c) (BONUS) Use the result of (b1) to derive the explicit formula for the Fibonacci numbers.

10. (2+4+10+3+15 points)

   (a) Define the norm of a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ under the standard dot product.

   (b) Define the operator norm $\|A\|$ of an $n \times n$ real matrix A.

   (c) Prove: if $B = (\beta_{i,j})$ is an $n \times n$ matrix then $|\beta_{i,j}| \leq \|B\|$.

   (d) Prove: if B is an $n \times n$ matrix then $B^T B$ is symmetric.

(e) Recall that Rayleigh's principle says that the largest eigenvalue of a symmetric real matrix A is the maximum of the Rayleigh quotient $R_A(x) = x^T Ax/x^T x$ ($x \in \mathbb{R}^n$, $x \neq 0$). Prove that the operator norm of a (not necessarily symmetric) $n \times n$ real matrix B is $\sqrt{\lambda}$ where λ is the largest eigenvalue of $B^T B$.

11. (8+8+6+8 points)

(a) Draw the diagram of a finite Markov Chain which has more than one stationary distribution. Give two stationary distributions for your Markov Chain. Use as few states as possible.

(b) Recall that a finite Markov Chain is *irreducible* if its transition digraph is strongly connected. Draw the diagram of a reducible (not irreducible) finite Markov Chain with a unique stationary distribution. State the stationary distribution. Do not prove. Use as few states as possible.

(c) Define ergodicity of a finite Markov Chain. Define the terms used in the definition in terms of directed graph concepts.

(d) Draw the diagram of an irreducible but non-ergodic finite Markov Chain.

12. (15+10 points)

(a) Consider the simple random walk on the integers: $X_0 = 0$ and $X_{t+1} = X_t \pm 1$, each possibility having probability 1/2. (The frog flips a coin at each step to decide whether to move right or left by one step.) Compute the probability that $X_{2n} = 0$ (in $2n$ steps the frog is back at the starting point). Give a simple closed-form expression.

(b) Asymptotically evaluate this probability. Give a very simple expression; no factorials or binomial coefficients.

13. (2+5+10+18+B8 points) Let $V = \{1, 2, \ldots, n\}$, $n \geq 3$. Let us consider a random graph $\mathcal{G}$ on the vertex set V; adjacency is decided by coin flips.

(a) What is the size of the sample space for this experiment?

(b) Given $m \geq 0$, what is the probability that $\mathcal{G}$ has exactly m edges?

(c) Let A_i denote the event that vertex i has even degree. What is the probability of A_i? (Prove.)

(d) Decide whether A_1 and A_2 are positively or negatively correlated or independent.

(e) (BONUS) What is the probability that all vertices of $\mathcal{G}$ have even degree?

14. (2+8+22 points) A careless secretary puts n distinct letters into n addressed envelopes at random. All addresses are different. Let X denote the number of letters that happen to get in the right envelope.

   (a) What is the size of the sample space of this experiment?

   (b) Determine $E(X)$. Clearly define your random variables.

   (c) Determine the probability that $X = 0$ (none of the letters goes in the right envelope). Name the method used. Prove that this probability approaches 1/e as $n \to \infty$.

15. (7B points)(BONUS) Prove: for almost all graphs G, $\chi(G) > \omega(G)^{100}$. ($\chi(G)$ denotes the chromatic number and $\omega(G)$ the clique number, i. e., the size of the largest clique (complete subgraph) in G.)

16. (9B points)(BONUS) Let $A = (a_{i,j})$ be a symmetric $n \times n$ real matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j}^2 = \sum_{i=1}^{n} \lambda_i^2 .$$

17. (6B points)(BONUS) Prove: there are infinitely many values of $k \geq 1$ such that $2^k + 1$ and $3^k + 1$ are not relatively prime.