CMSC-37110 Discrete Mathematics
MIDTERM EXAM October 27, 2008

This exam contributes 16% to your course grade.

Do **not** *use book, notes.* You **may** use a calculator for *basic arithmetic* but not for more advanced functions such as g.c.d. **Show all your work.** If you are not sure of the meaning of a problem, **ask the proctor.** The *bonus problems* are underrated, do not work on them until you are done with everything else.

1. (12 points) Decide whether or not the following system of congruences is solvable. Prove your answer; do not solve.

   $x \equiv 13 \pmod{21}$

   $x \equiv 8 \pmod{15}$

   $x \equiv 3 \pmod{5}$

2. (18 points) Let p be a prime number and let $f(x) = 1+x+x^2+\cdots+x^{p-2}$. Prove: $(\forall x)(f(x) \equiv -1, 0 \text{ or } 1 \pmod{p})$.

3. (6+14 points) (a) Count the increasing functions $f : [k] \to [n]$. (Recall the notation $[k] = \{1, \ldots, k\}$.) (b) Count those functions $f : [k] \to [n]$ that satisfy $f(i+1) \geq f(i) + 2$ for every i. Your answers should be simple expressions involving binomial coefficients. Prove your answers.

4. (8+10 points) Give closed-form expressions for the generating functions of the sequences (a) $a_n = n$ and (b) $b_n = n^2$.

5. (8 points) Let Ω be a sample space with a prime number of elements, and consider the uniform distribution over Ω. Prove that in this probability space, no two nontrivial events are independent.

6. (12 points) Let $\pi(n)$ denote the number of primes $\leq n$. Recall that the Prime Number Theorem states that $\pi(n) \sim n/\ln n$. True or false: $\pi(n) = O(n^{0.9})$? Prove your answer.

7. (8 points) Determine the coefficient of $x^5y^2z^3$ in the expansion of $(x+y+z)^{10}$. Show all your work; do not use calculator. Write your answer as a product of prime powers (like $2^4 \times 5^2 \times 11$).

8. (8+8+8+8B points) We roll n (6-sided) dice. (a) Let $p(n,k)$ denote the probability that exactly k of them show the number 6. Determine $p(n,k)$. Give a simple closed-form expression involving a binomial coefficient. (b) What is the expected number of dice that show "6"? (Define your variables!) (c) What is the expected value of the product of all numbers shown? (d) (BONUS) Fix n. Show that the value $p(n,k)$ increases for a while (as a function of k) and then decreases. Where is the turning point (the most likely value of k)? Prove all your answers.

9. (5+5 points) (a) Find a sequence a_n such that $\lim_{n\to\infty} a_n = 1$ but $\lim_{n\to\infty} a_n^n = \infty$. (b) Find a sequence b_n such that $\lim_{n\to\infty} b_n = \infty$ but $b_{n+1} \sim b_n$.

10. (6+6 points) (a) Count the graphs on vertex set $V = [n]$. (b) How many among these will have exactly m edges? Your answers should be simple closed-form expressions involving binomial coefficients.

11. (8 points) Let G be a graph and $\overline{G}$ its complement. Prove: if G is disconnected then $\overline{G}$ is connected.

12. (10 points) Prove: if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$ and $a_n = \Theta(b_n)$ then $\ln a_n \sim \ln b_n$.

13. (BONUS 10B points) Let q_n be the probability that a random string of length n over the alphabet $\{A, B\}$ does not contain consecutive As. Asymptotically evaluate q_n. Your answer should be of the form ab^n. Determine a and b.

14. (BONUS 10B points) Let p be a prime and $t \geq 1$. Suppose p^t divides the binomial coefficient $\binom{n}{k}$. Prove: $p^t \leq n$.

15. (BONUS 6B points) Let $G = (V, E)$ be a graph. Prove: it is possible to split V into two disjoint subsets A and B ($V = A \dot\cup B$) such that at least half the edges of G go between A and B.