CMSC-37110 Discrete Mathematics
SECOND QUIZ October 17, 2008

Name (print): ________________________________

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. **Write your solution in the space provided.** You may continue on the reverse. This exam contributes 6% to your course grade.

All variables in the problems below are <u>integers</u>.

1. (4 points) **True** or **false** (circle one, prove): $(\forall x, y)(\exists z)\,(\gcd(x,z) + \gcd(y,z) = \gcd(x+y,z))$.

2. (10 points) Prove: $\ln(1 + 1/n) \sim 1/n$.

3. (6+14 points) Count the (a) increasing (b) nondecreasing functions $f : [k] \to [n]$. (Recall the notation $[k] = \{1, \ldots, k\}$.) Your answers should be simple expressions involving binomial coefficients. Prove your answers.

4. (7+7 points) Give a closed-form expression (no summation symbols, no dot-dot-dots) for each sum: (a) $\sum_{k=0}^{n}(2/3)^k$; (b) $\sum_{k=0}^{n}\binom{n}{k}(2/3)^k$

5. (12 points) Find values a, b, c such that the sequence x_n defined by the recurrence $x_n = a + bx_{n-1}$ with initial value $x_0 = c$ does not have a limit (either finite or infinite). Indicate why your sequence has no limit.

6. BONUS (2B points) Prove:

$$\prod_{n/2<p\leq n} p < 2^n.$$

(The product is over all primes in the range $(n/2, n]$.)

7. BONUS (3B points) Prove: Prove: the product of any k consecutive integers is divisible by $k!$. (Hint: one-line proof.)

8. BONUS (6B points) True or false? Prove your answer.

$$\left(1+\frac{1}{n}\right)^{n^2} \sim \mathrm{e}^n.$$