CMSC-37110 Discrete Mathematics
THIRD QUIZ November 10, 2008

Name (print): ______________________________

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. **Write your solution in the space provided.** You may continue on the reverse. This exam contributes 6% to your course grade.

1. (3+3+3+3+3 points) What is wrong with each of these statements? Explain, fix. Each explanation should be a very short but informative (bug-specific, not generic) phrase. The fix should make minimum change to the original statement. (a) $\lim_{n\to\infty}(n^2+n) = n^2$. (b) $\lim_{n\to\infty} 2^n \to \infty$. (c) $\pi(x) \sim n/\ln n$. (d) If X and Y are independent events then $E(X+Y) = E(X)+E(Y)$. (e) In Euler's "$n-m+r=2$" equation, r is the number of regions of a planar graph.

2. (4+6 points) Prove: (a) $\binom{2n}{n} < 4^n$. (b) $\binom{2n}{n} > \frac{4^n}{2n+1}$. (Do not use Stirling's formula. Each proof should be a one-liner.)

3. (10 points) Prove: $n^{100} = o(1.01^{\sqrt{n}})$ (little-oh). Use the fact that $\ln x = o(x)$. Beyond this, use only algebra and basic facts about logarithms and limits. Do not use l'Hôpital's rule.

4. (10 points) Give two plane drawings of a simple connected planar graph such that in one of the drawings there is a three-sided region and in the other there is no three-sided region. State the number of sides of each region in each of your drawings. Your graph should have as few vertices and edges as possible, but do not prove minimality.

5. (1+4+10 points) Let us pick a random string w of length n over the alphabet $\{A, B\}$. (a) What is the size of the sample space for this experiment? (b) Let X denote the number of occurrences of the substring AA in w. (For instance, if $w = AAABBAA$ then $X = 3$.) Determine $E(X)$. (c) Determine $\mathrm{Var}(X)$. Give a very simple closed-form expression. Make sure you define your random variables.

6. BONUS (3B points) Let us say that a graph is *lean* if $m \le n + 2$. Let us say that a graph is *hereditarily lean* if the same holds for every subgraph. Prove: every hereditarily lean graph is planar.

7. BONUS (4B points) Prove: almost all graphs are not regular. (A graph is regular if all vertices have the same degree.)

8. BONUS (4B points) True or false? Prove your answer.

$$\left(1 + \frac{1}{n}\right)^{n^2} \sim \mathrm{e}^n.$$