CMSC-37110 Discrete Mathematics
FOURTH QUIZ December 1, 2008

Name (print): ________________________________

Do **not** *use book, notes, scratch paper.* **Show all your work.** If you are not sure of the meaning of a problem, **ask the instructor.** The *bonus problems* are underrated, do not work on them until you are done with everything else. **Write your solution in the space provided.** You may continue on the reverse. This exam contributes 6% to your course grade.

1. (2+2+2+2 points) Fill in the blanks. An $n \times n$ matrix A is invertible if and only if (a) its rank is ________ (b) its determinant is ________ (c) its eigenvalues are ________ (d) the system $Ax = 0$ of equations has ____________________ solution(s).

2. (6 points) Give a 3×3 matrix $A \neq I$ (not the identity matrix) with characteristic polynomial $f_A(t) = (t-1)^3$. Indicate why your matrix is right.

3. (8 points) Describe, in terms of elementary geometry, a linear transformation of the plane which has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. Do not use coordinates in your description. Your complete description should be no more than one line. Do not prove.

4. (2+8+4 points) Consider the space $V = \mathbb{R}^{(n)}[x]$ of polynomials of degree $\leq n$ and the transformation $\varphi : V \to V$ defined by $\varphi(f) = xf'$.
(a) State a basis of V and state $\dim(V)$. (b) Compute the matrix of φ with respect to this basis. (c) Determine the eigenvalues of φ.

5. (12 points) Determine the rank of the $n \times n$ matrix $B = (\beta_{i,j})$ of which the entries are $\beta_{i,j} = i + j$.

6. (12 points) How many diagonal matrices have the characteristic polynomial $f(t) = (t-1)^7(t-5)^3(t-6)^8$? Your answer should be a simple closed-form expression. Do not evaluate. Indicate why your answer is correct.

7. BONUS (4B points) Let G be a bipartite graph with adjacency matrix A. Prove: if λ is an eigenvalue of A then $-\lambda$ is also an eigenvalue of A. Prove this using the definition of eigenvalues only; do not refer to a theorem from class.

8. BONUS (6B points) Prove: if A is a stochastic matrix then $|\det(A)| \leq 1$.

9. BONUS (12B points) Prove: if all (complex) eigenvalues of an $n \times n$ matrix A have absolute value less than 1 then $\lim_{k\to\infty} A^k = 0$.