1. (2+2+2+2 points) Fill in the blanks. An \( n \times n \) matrix \( A \) is invertible if and only if (a) its rank is ______ (b) its determinant is ______ (c) its eigenvalues are ______ (d) the system \( Ax = 0 \) of equations has __________ solution(s).

2. (6 points) Give a \( 3 \times 3 \) matrix \( A \neq I \) (not the identity matrix) with characteristic polynomial \( f_A(t) = (t - 1)^3 \). Indicate why your matrix is right.

3. (8 points) Describe, in terms of elementary geometry, a linear transformation of the plane which has eigenvalues \( \lambda_1 = 1 \) and \( \lambda_2 = -1 \). Do not use coordinates in your description. Your complete description should be no more than one line. Do not prove.

4. (2+8+4 points) Consider the space \( V = \mathbb{R}^{(n)}[x] \) of polynomials of degree \( \leq n \) and the transformation \( \varphi : V \rightarrow V \) defined by \( \varphi(f) = xf' \).
   (a) State a basis of \( V \) and state \( \dim(V) \). (b) Compute the matrix of \( \varphi \) with respect to this basis. (c) Determine the eigenvalues of \( \varphi \).
5. (12 points) Determine the rank of the $n \times n$ matrix $B = (\beta_{i,j})$ of which the entries are $\beta_{i,j} = i + j$.

6. (12 points) How many diagonal matrices have the characteristic polynomial $f(t) = (t - 1)^7(t - 5)^3(t - 6)^8$? Your answer should be a simple closed-form expression. Do not evaluate. Indicate why your answer is correct.

7. BONUS (4B points) Let $G$ be a bipartite graph with adjacency matrix $A$. Prove: if $\lambda$ is an eigenvalue of $A$ then $-\lambda$ is also an eigenvalue of $A$. Prove this using the definition of eigenvalues only; do not refer to a theorem from class.

8. BONUS (6B points) Prove: if $A$ is a stochastic matrix then $|\det(A)| \leq 1$.

9. BONUS (12B points) Prove: if all (complex) eigenvalues of an $n \times n$ matrix $A$ have absolute value less than 1 then $\lim_{k \to \infty} A^k = 0$. 